Spectral methods for orthogonal rational functions

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Abstract

An operator theoretic approach to orthogonal rational functions on the unit circle with poles in its exterior is presented in this paper. This approach is based on the identification of a suitable matrix representation of the multiplication operator associated with the corresponding orthogonality measure. Two different alternatives are discussed, depending whether we use for the matrix representation the standard basis of orthogonal rational functions, or a new one with poles alternatively located in the exterior and the interior of the unit circle. The corresponding representations are linear fractional transformations with matrix coefficients acting respectively on Hessenberg and five-diagonal unitary matrices.

In consequence, the orthogonality measure can be recovered from the spectral measure of an infinite unitary matrix depending uniquely on the poles and the parameters of the recurrence relation for the orthogonal rational functions. Besides, the zeros of the orthogonal and para-orthogonal rational functions are identified as the eigenvalues of matrix linear fractional transformations of finite Hessenberg and five-diagonal matrices.

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As an application of this operator approach, we obtain new relations between the support of the orthogonality measure and the location of the poles and parameters of the recurrence relation, generalizing to the rational case known results for orthogonal polynomials on the unit circle.

Finally, we extend these results to orthogonal polynomials on the real line with poles in the lower half plane.

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1 Introduction

The connection with Jacobi matrices has led to numerous applications of spectral techniques for self-adjoint operators in the theory of orthogonal polynomials on the real line. The direct extension of these ideas to the orthogonal polynomials on the unit circle yields a connection with unitary Hessenberg matrices (see [3, 15, 20, 30, 33]) which has provided some applications (see for instance [16, 17, 18, 19, 33]). Nevertheless, the authentic analogue of the Jacobi matrices for the unit circle is a class of unitary five-diagonal matrices which has been only recently discovered (see [36, 12]). This discovery has caused an explosion of applications of spectral methods for unitary operators in the theory of orthogonal polynomials on the unit circle, among which the numerous applications appearing in the monograph [30, 31] have been only the starting point.

The orthogonal polynomials are a particular case of a more general kind of orthogonal functions with interest in many pure and applied sciences: the orthogonal rational functions with prescribed poles (see [10] and the references therein). The natural generalization of the orthogonal polynomials on the real line and the unit circle requires the poles to be in the extended real line and in the exterior of the closed unit disk respectively. The first situation presents special complications, an indication of this being the fact that the poles can lie on the support of the orthogonality measure. Indeed, considered as orthogonal rational functions, the main difference between the orthogonal polynomials on the real line and the unit circle is not the location

of the support of the measure, but the relative location of the poles with respect to this support. Actually, the Cayley transform maps the orthogonal rational functions on the unit circle with poles in the exterior of the closed unit disk onto the orthogonal rational functions on the real line with poles in the lower half plane, so both of them can be thought as generalizations of the orthogonal polynomials on the unit circle. The purpose of the paper is to generalize to this kind of orthogonal rational functions the above referred spectral techniques for the orthogonal polynomials on the unit circle.

An important ingredient in the theory of orthogonal rational functions are the linear fractional transformations $z \to (a_1 z + a_2)(a_3 z + a_4)^{-1}$ on the complex plane, where a_i are complex numbers. It is natural to expect the related spectral methods to have a close relationship with the operator version of such transformations, i.e., the maps $T \to (A_1 T + A_2)(A_3 T + A_4)^{-1}$ in the space of linear operators on a Hilbert space, where the coefficients A_i are now operators on the same Hilbert space. The theory of linear fractional transformations with operator coefficients goes back to the work [25] of M. G. Krein and Yu. L. Šmuljan, motivated by the study of operators in spaces with an indefinite metric initiated by M. G. Krein in [23, 24]. As we will see, the matrices related to the rational analogue of the orthogonal polynomials on the unit circle are the result of applying a linear fractional transformation with matrix coefficients to the Hessenberg and five-diagonal unitary matrices associated with the polynomial case.

This reason, and also a better understanding of the subsequent rational generalizations, motivates Section 2, which summarizes the basics on spectral methods for orthogonal polynomials on the unit circle and describes the main results needed about orthogonal rational functions on the unit circle with poles in the exterior of the closed unit disk. Section 3 introduces the operator linear fractional transformations of interest for such orthogonal rational functions. The corresponding spectral theory is developed in Sections 4 and 5, which are devoted to the approaches based on Hessenberg and five-diagonal matrices respectively. Section 6 presents some applications of the above spectral theory to the study of the relation between the support of the orthogonality measure and the poles and parameters of the recurrence relation for the orthogonal rational functions. Finally, the Appendix remarks the main analogies and differences with the spectral theory for orthogonal rational functions on the real line with poles lying on the lower half plane.

2 OP and ORF on the unit circle

In what follows a measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ will be a probability Borel measure μ supported on a subset suppμ of $\mathbb{T}$. Let μ be one of such measures and consider the Hilbert space L^2_μ of μ-square-integrable functions with inner product

$$\langle f, g \rangle_\mu = \int \overline{f(z)} g(z)\, d\mu(z), \qquad f, g \in L^2_\mu.$$

Unless we say the opposite we will suppose that suppμ is an infinite set. Then, $(z^n)_{n\geq 0}$ is a linearly independent subset of L^2_μ whose orthonormalization gives the orthogonal polynomials (OP) $(\varphi_n)_{n\geq 0}$ with respect to μ. If we choose these polynomials with positive leading coefficient, they satisfy the recurrence relation

$$\begin{gathered} \varphi_0 = 1, \\ \rho_n \begin{pmatrix} \varphi_n \\ \varphi_n^* \end{pmatrix} = \begin{pmatrix} 1 & a_n \\ \overline{a}_n & 1 \end{pmatrix} \begin{pmatrix} z\varphi_{n-1} \\ \varphi_{n-1}^* \end{pmatrix}, \qquad n \geq 1, \\ a_n = \frac{\varphi_n(0)}{\varphi_n^*(0)}, \qquad \rho_n = \sqrt{1 - |a_n|^2}, \end{gathered} \tag{1}$$

where $\varphi_n^*(z) = z^n \overline{\varphi}_n(1/z)$ and $|a_n| < 1$. This establishes a bijection between measures μ on $\mathbb{T}$ and sequences $(a_n)_{n\geq 1}$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

A central problem in the theory of OP on the unit circle is to find relations between the orthogonality measure μ and the sequence $(a_n)_{n\geq 1}$ appearing in the recurrence relation for the OP. There are several approaches to this problem but these last years have seen a rapid and impressive development of new operator theory techniques (see [30, 31, 32] and references therein) based on the recent discovery of the analogue for the unit circle of the Jacobi matrix related to OP on the real line (see [12, 36]).

The main tool for the operator theoretic approach to the OP on $\mathbb{T}$ is the unitary multiplication operator

$$\begin{aligned} T_\mu \colon L^2_\mu &\to L^2_\mu \\ f(z) &\to z f(z) \end{aligned}$$

It is known that the spectrum of T_μ coincides with suppμ and the eigenvalues of T_μ, which have geometric multiplicity 1, are the mass points of μ. The eigenvectors of a given eigenvalue λ are spanned by the characteristic function

$\mathcal{X}_{\{\lambda\}}$ of the set $\{\lambda\}$. Even more, if E is the spectral measure of T_μ then $\mu(\Delta) = \langle 1, E(\Delta)1\rangle_\mu$ for any Borel subset Δ of $\mathbb{T}$. All these properties are true no matter whether suppμ is finite or infinite.

If $(f_n)_{n\geq 0}$ is a basis of L^2_μ, the matrix of T_μ with respect to $(f_n)_{n\geq 0}$ is the matrix M whose (i,j)-th element is $M_{ij} = \langle f_i, T_\mu f_j\rangle_\mu$. In other words,

$$\begin{pmatrix} zf_0(z) & zf_1(z) & \cdots \end{pmatrix} = \begin{pmatrix} f_0(z) & f_1(z) & \cdots \end{pmatrix} M. \tag{2}$$

Any matrix representation M of T_μ can be identified with the unitary operator

$$\begin{array}{c} \ell^2 \to \ell^2 \\ {\scriptstyle x \to Mx} \end{array}$$

on the space ℓ^2 of square-sumable complex sequences. This operator is unitarily equivalent to T_μ. Therefore, once we know the dependence of M on the parameters $(a_n)_{n\geq 1}$, this matrix permits us to recover the orthogonality measure μ starting from the recurrence relation of the OP. Regarding this problem, the utility of the matrix representation M depends on its simplicity as a function of the parameters $(a_n)_{n\geq 1}$.

For instance, when the polynomials are dense in L^2_μ, the representation of T_μ with respect to the OP $(\varphi_n)_{n\geq 0}$ is the irreducible Hessenberg matrix (see [3, 15, 20, 30, 33])

$$\mathcal{H} = \begin{pmatrix} -a_1 & -\rho_1 a_2 & -\rho_1\rho_2 a_3 & -\rho_1\rho_2\rho_3 a_4 & \cdots \\ \rho_1 & -\overline{a}_1 a_2 & -\overline{a}_1\rho_2 a_3 & -\overline{a}_1\rho_2\rho_3 a_4 & \cdots \\ 0 & \rho_2 & -\overline{a}_2 a_3 & -\overline{a}_2\rho_3 a_4 & \cdots \\ 0 & 0 & \rho_3 & -\overline{a}_3 a_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{3}$$

$\mathcal{H} = (h_{i,j})$ is called a Hessenberg matrix because $h_{i,j} = 0$ for $i > j+1$, and the irreducibility means that $h_{j+1,j} \neq 0$ for any j. Using the 2×2 symmetric unitary matrices

$$\Theta_n = \begin{pmatrix} -a_n & \rho_n \\ \rho_n & \overline{a}_n \end{pmatrix}, \qquad n \geq 1, \tag{4}$$

the Hessenberg representation can be factorized as

$$\mathcal{H} = \lim_n \begin{pmatrix} \Theta_1 & \\ & I \end{pmatrix} \begin{pmatrix} I_1 & & \\ & \Theta_2 & \\ & & I \end{pmatrix} \begin{pmatrix} I_2 & & \\ & \Theta_3 & \\ & & I \end{pmatrix} \cdots \begin{pmatrix} I_{n-1} & & \\ & \Theta_n & \\ & & I \end{pmatrix}, \tag{5}$$

where I and I_n mean the identity matrix of order infinite and n respectively and the limit has to be understood in the strong sense.

Apart from its complexity, the Hessenberg representation has the inconvenience of being valid only when the polynomials are dense in L^2_μ. In the general case $\mathcal{H}$ is a matrix representation of the restriction $T_\mu \restriction \mathcal{P} \colon \mathcal{P} \to \mathcal{P}$ of T_μ to the T_μ-invariant subspace given by the closure $\mathcal{P}$ of the polynomials in L^2_μ. As a restriction of a unitary operator, $T_\mu \restriction \mathcal{P}$ is isometric but not necessarily unitary. $\mathcal{H}$ is a representation of T_μ iff any of the following equivalent conditions hold (see [15, 30]):

$$\mathcal{P} = L^2_\mu \Leftrightarrow \log \mu' \notin L^1_m \Leftrightarrow (a_n)_{n\geq 1} \notin \ell^2 \Leftrightarrow \mathcal{H} \text{ is unitary.}$$

We denote by m the Lebesgue measure on $\mathbb{T}$.

A way to avoid the problems of the Hessenberg representation is to use as a basis of L^2_μ the Laurent OP $(\chi_n)_{n\geq 0}$ that arise from the orthonormalization of $(1, z, z^{-1}, z^2, z^{-2}, \dots)$, which are given by (see [12, 30, 34, 36])

$$\chi_{2n}(z) = z^{-n}\varphi^*_{2n}(z), \qquad \chi_{2n+1}(z) = z^{-n}\varphi_{2n+1}(z), \qquad n \geq 0.$$

The corresponding representation of T_μ is the five-diagonal matrix (see [12, 30, 36])

$$\mathcal{C} = \begin{pmatrix} -a_1 & -\rho_1 a_2 & \rho_1\rho_2 & 0 & 0 & 0 & 0 & \cdots \\ \rho_1 & -\overline{a}_1 a_2 & \overline{a}_1\rho_2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\rho_2 a_3 & -\overline{a}_2 a_3 & -\rho_3 a_4 & \rho_3\rho_4 & 0 & 0 & \cdots \\ 0 & \rho_2\rho_3 & \overline{a}_2\rho_3 & -\overline{a}_3 a_4 & \overline{a}_3\rho_4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\rho_4 a_5 & -\overline{a}_4 a_5 & -\rho_5 a_6 & \rho_5\rho_6 & \cdots \\ 0 & 0 & 0 & \rho_4\rho_5 & \overline{a}_4\rho_5 & -\overline{a}_5 a_6 & \overline{a}_5\rho_6 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\rho_6 a_7 & -\overline{a}_6 a_7 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad (6)$$

which, apart from being valid for any measure μ on $\mathbb{T}$, is a band instead of a Hessenberg matrix. Also, it has a much simpler dependence on the parameters $(a_n)_{n\geq 1}$ than in the Hessenberg case. Moreover, this five-diagonal representation has a much better factorization than the Hessenberg one since $\mathcal{C} = \mathcal{C}_o\mathcal{C}_e$, where $\mathcal{C}_o$ and $\mathcal{C}_e$ are the 2×2-block-diagonal symmetric unitary matrices

$$\mathcal{C}_o = \begin{pmatrix} \Theta_1 & & & \\ & \Theta_3 & & \\ & & \Theta_5 & \\ & & & \ddots \end{pmatrix}, \qquad \mathcal{C}_e = \begin{pmatrix} I_1 & & & \\ & \Theta_2 & & \\ & & \Theta_4 & \\ & & & \ddots \end{pmatrix}. \qquad (7)$$

Alternatively, it is possible to orthonormalize $(1, z^{-1}, z, z^{-2}, z^2, \ldots)$. This leads to the Laurent OP $(\chi_{n*})_{n\geq 0}$ where $\chi_{n*}(z) = \overline{\chi}_n(1/z)$, i.e.,

$$\chi_{2n*}(z) = z^{-n}\varphi_{2n}(z), \qquad \chi_{2n+1*}(z) = z^{-n-1}\varphi^*_{2n+1}(z), \qquad n \geq 0.$$

The related representation of T_μ is simply the transposed matrix $\mathcal{C}^T = \mathcal{C}_e\mathcal{C}_o$ of $\mathcal{C}$.

The Hessenberg and five-diagonal matrices given in (3) and (6) represent a multiplication operator (on $\mathcal{P}$ or L^2_μ) only when $(a_n)_{n\geq 1}$ lies on $\mathbb{D}$. Nevertheless, they are well defined matrices for any sequence $(a_n)_{n\geq 1}$ in the closed unit disk $\overline{\mathbb{D}}$. Indeed, factorizations (5) and (7) show that, even in this case, the Hessenberg representation is isometric while the five-diagonal one is unitary. Furthermore, when some $a_n \in \mathbb{T}$, these Hessenberg and five-diagonal matrices decompose as a direct sum of an $n \times n$ and an infinite matrix. This decomposition property is of interest when trying to make perturbative spectral analysis of such matrix representations.

The Hessenberg and five-diagonal representations of T_μ also give a spectral interpretation for the zeros of the OP in terms of the parameters of the recurrence. This result comes from the relation between the OP and certain orthogonal truncations of the operator T_μ. The restriction $T_\mu \restriction \mathcal{P}_{n,l}$ of the multiplication operator T_μ to the subspace $\mathcal{P}_{n,l} = \operatorname{span}\{z^l, z^{l+1}, \ldots, z^{l+n-1}\}$ has no sense since $\mathcal{P}_{n,l}$ is not invariant under T_μ. To give sense to this kind of restriction we must multiply T_μ on the left by a projection on $\mathcal{P}_{n,l}$. In particular, if $P_{n,l}\colon L^2_\mu \to L^2_\mu$ is the orthogonal projection on $\mathcal{P}_{n,l}$, the operator $T_\mu^{(\mathcal{P}_{n,l})} = P_{n,l}T_\mu \restriction \mathcal{P}_{n,l}$ is called the orthogonal truncation of T_μ on $\mathcal{P}_{n,l}$. The key point is that, for any $l \in \mathbb{Z}$, the characteristic polynomial of $T_\mu^{(\mathcal{P}_{n,l})}$ is, up to factors, the n-th OP φ_n (see [30]).

The first n OP $(\varphi_k)_{k=0}^{n-1}$ are a basis of $\mathcal{P}_n = \mathcal{P}_{n,0}$ and the related matrix of $T_\mu^{(\mathcal{P}_n)}$ is the principal submatrix $\mathcal{H}_n$ of $\mathcal{H}$ of order n. So, φ_n is proportional to the characteristic polynomial of $\mathcal{H}_n$, whose eigenvalues are therefore the zeros of φ_n. Furthermore, for $l = -[(n-1)/2]$, the first n Laurent OP $(\chi_k)_{k=0}^{n-1}$ are a basis of $\mathcal{P}_{n,l}$ and the corresponding matrix of $T_\mu^{(\mathcal{P}_{n,l})}$ is the principal submatrix $\mathcal{C}_n$ of $\mathcal{C}$ of order n. Hence, φ_n is proportional to the characteristic polynomial of $\mathcal{C}_n$ and, thus, the zeros of φ_n are the eigenvalues of $\mathcal{C}_n$.

Contrary to the full infinite matrix, $\mathcal{H}_n$ and $\mathcal{C}_n$ are not unitary and depend only on the first n parameters $a_1, \ldots, a_n$. However, factorizations (5) and (7) show that if we change in these principal submatrices the last parameter $a_n \in \mathbb{D}$ by a complex number $u \in \mathbb{T}$, then we obtain a unitary matrix. The

corresponding characteristic polynomial is the result of performing n steps of recurrence (1), but substituting in the last one $a_n \in \mathbb{D}$ by $u \in \mathbb{T}$, i.e., it is a multiple of

$$z\varphi_{n-1}(z) + u\varphi_{n-1}^*(z).$$

Using (1), this polynomial can be alternatively written up to factors as

$$\varphi_n(z) + v\varphi_n^*(z), \qquad v = \frac{u-a}{1-\overline{a}_n u},$$

and the arbitrariness of $u \in \mathbb{T}$ translates into a similar arbitrariness for $v \in \mathbb{T}$. These polynomials, called para-orthogonal polynomials (POP), were introduced for the first time in [21]. There it was proved that such POP have simple zeros lying on $\mathbb{T}$, which play the role of nodes in the Szegő quadrature formulas on $\mathbb{T}$ (the analogue of the Gaussian quadrature formulas on $\mathbb{R}$), thus, providing finitely supported measures on $\mathbb{T}$ that $*$-weakly converge to the measure μ. Therefore, the nodes of the Szegő quadrature formulas can be obtained as eigenvalues of Hessenberg or five-diagonal unitary matrices.

Our aim is to generalize the above results to the orthogonal rational functions with poles outside of the support of the orthogonality measure. Two archetypical situations will be considered: measures on the unit circle $\mathbb{T}$ and measures on the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. For convenience, the analysis will be done in a detailed way for measures on the unit circle, the discussion of the special features in the case of the real line being relegated to the Appendix. So, for the moment we will consider a measure μ on $\mathbb{T}$ and the corresponding orthogonal rational functions with poles arbitrary located in the exterior of the unit circle $\mathbb{E} = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. We consider the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to include for the poles the possibility of being located at ∞. Indeed, the OP with respect to μ correspond to the special case of the orthogonal rational functions with all the poles at ∞.

An important transformation in $\overline{\mathbb{C}}$ is $\hat{z} = 1/\overline{z}$, which leaves invariant any element of $\mathbb{T}$ and establishes a bijection between $\mathbb{D}$ and $\mathbb{E}$. This transformation induces the $*$-involution $f_*(z) = \overline{f(\hat{z})}$ in the set of complex functions, which defines an anti-unitary operator on L^2_μ for any measure μ on $\mathbb{T}$. As a consequence, a sequence $(f_n)_{n\geq 0}$ of functions is a basis of L^2_μ iff $(f_{n*})_{n\geq 0}$ is a basis too. Moreover, the $*$-involution on (2) gives

$$\begin{pmatrix} zf_{0*}(z) & zf_{1*}(z) & \cdots \end{pmatrix} = \begin{pmatrix} f_{0*}(z) & f_{1*}(z) & \cdots \end{pmatrix} \overline{M}^{-1},$$

which, taking into account that M is unitary, shows that the matrix of T_μ with respect to $(f_{n*})_{n\geq 0}$ is the transposed M^T of the matrix M associated with $(f_n)_{n\geq 0}$. This relation holds when μ is finitely supported too, with the only difference that the basis of L^2_μ are finite.

Another essential ingredient in the theory of orthogonal rational functions on $\mathbb{T}$ are the Möbius transformations ζ_α defined for any $\alpha \in \mathbb{D}$ by

$$\zeta_\alpha(z) = \frac{\varpi^*_\alpha(z)}{\varpi_\alpha(z)}, \qquad \begin{cases} \varpi_\alpha(z) = 1 - \overline{\alpha} z, \\ \varpi^*_\alpha(z) = z\varpi_{\alpha*}(z) = z - \alpha. \end{cases}$$

Up to factors in $\mathbb{T}$, they are all the automorphisms of $\mathbb{D}$. Indeed, ζ_α is a bijection of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ that leaves invariant $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{E}$. The inverse transformation of ζ_α is $\tilde{\zeta}_\alpha = \zeta_{-\alpha}$. It is also remarkable that $\zeta_{\alpha*} = 1/\zeta_\alpha$. We distinguish the value $\alpha_0 = 0$ that gives $\zeta_{\alpha_0}(z) = z$.

To get rational functions with fixed poles in $\mathbb{E}$ we introduce a sequence $(\alpha_n)_{n\geq 1}$ in $\mathbb{D}$. This sequence defines the finite Blaschke products $(B_n)_{n\geq 0}$ given by

$$\begin{aligned} &B_0 = 1, \\ &B_n = \zeta_{\alpha_1} \cdots \zeta_{\alpha_n}, \qquad n \geq 1. \end{aligned} \tag{8}$$

Notice that $B_{n*} = 1/B_n$. The subspace

$$\mathcal{L}_n = \operatorname{span}\{B_0, B_1, \ldots, B_{n-1}\} = \frac{\mathcal{P}_n}{\varpi_{\alpha_1} \cdots \varpi_{\alpha_{n-1}}}$$

consists of those rational functions whose poles, counted with multiplicity, lie on $(\hat{\alpha}_k)_{k=1}^{n-1}$. We use the notation $\mathcal{L}_\infty = \operatorname{span}\{B_n\}_{n\geq 0} = \cup_{n\geq 1}\mathcal{L}_n$ for the set of rational functions with poles lying on $(\hat{\alpha}_n)_{n\geq 1}$, counted with multiplicity, and $\mathcal{L}$ for the closure of $\mathcal{L}_\infty$ in L^2_μ.

If μ is a measure on $\mathbb{T}$ we can consider the rational functions $(\Phi_n)_{n\geq 0}$ that arise from the orthonormalization of $(B_n)_{n\geq 0}$ in L^2_μ. $(\Phi_n)_{n\geq 0}$ are called orthogonal rational functions (ORF) with respect to μ associated with $(\alpha_n)_{n\geq 1}$. When referring to $(\Phi_n)_{n\geq 0}$ we will call it in short a sequence of ORF on the unit circle. These functions satisfy a recurrence relation which, with an appropriate normalization of $(\Phi_n)_{n\geq 0}$, has the form (see [10, Theorem 4.1.3])

$$\begin{aligned} &\Phi_0 = 1, \\ &\begin{pmatrix} \Phi_n \\ \Phi^*_n \end{pmatrix} = e_n \frac{\varpi_{n-1}}{\varpi_n} \begin{pmatrix} 1 & b_n \\ \overline{b}_n & 1 \end{pmatrix} \begin{pmatrix} z_n \zeta_{n-1} \Phi_{n-1} \\ \Phi^*_{n-1} \end{pmatrix}, \qquad n \geq 1, \end{aligned} \tag{9}$$

where

$$b_n = \frac{\Phi_n(\alpha_{n-1})}{\Phi_n^*(\alpha_{n-1})}, \quad z_n = \begin{cases} -\frac{|\alpha_n|}{\alpha_n} & \text{if } \alpha_n \neq 0, \\ 1 & \text{if } \alpha_n = 0, \end{cases} \quad e_n = \sqrt{\frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})} \frac{1}{1-|b_n|^2}},$$

and we use the notation

$$\zeta_n = \zeta_{\alpha_n}, \quad \varpi_n = \varpi_{\alpha_n}, \quad \varpi_n^* = \varpi_{\alpha_n}^*, \quad \Phi_n^* = z_1 z_2 \cdots z_n B_n \Phi_{n*}.$$

Notice that we do not follow the standard notation $\zeta_n = z_n\zeta_{\alpha_n}$ and $B_n = z_1\zeta_{\alpha_1} \cdots z_n\zeta_{\alpha_n}$ (see for instance [10]). In fact, concerning the matrix representations of the multiplication operator, it is more convenient to avoid the presence of the factors z_n in recurrence (9), something that we can get using the ORF $(\phi_n)_{n\geq 0}$ given by

$$\begin{aligned} &\phi_0 = 1, \\ &\phi_n = \overline{z}_1 \overline{z}_2 \cdots \overline{z}_n \Phi_n, \qquad n \geq 1, \end{aligned}$$

and defining the superstar operation omitting the factors z_n, that is,

$$\phi_n^* = B_n \phi_{n*}.$$

Then, (9) is equivalent to

$$\begin{aligned} &\phi_0 = 1, \\ &\begin{pmatrix} \phi_n \\ \phi_n^* \end{pmatrix} = e_n \frac{\varpi_{n-1}}{\varpi_n} \begin{pmatrix} 1 & a_n \\ \overline{a}_n & 1 \end{pmatrix} \begin{pmatrix} \zeta_{n-1}\phi_{n-1} \\ \phi_{n-1}^* \end{pmatrix}, \qquad n \geq 1, \end{aligned} \tag{10}$$

with

$$a_n = \frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})} = \overline{z}_1 \overline{z}_2 \cdots \overline{z}_n b_n, \qquad e_n = \sqrt{\frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})} \frac{1}{1-|a_n|^2}}.$$

In the polynomial case, corresponding to $\alpha_n = 0$ for all n, (10) gives exactly (1). As in the polynomial situation, the parameters $(a_n)_{n\geq 1}$ of (10) lie on $\mathbb{D}$. A Favard-type theorem also holds (see [10, Theorem 8.1.4]): given a sequence $(a_n)_{n\geq 1}$ in $\mathbb{D}$, the functions $(\phi_n)_{n\geq 0}$ defined by recurrence (10) are orthonormal with respect to some measure on $\mathbb{T}$. This measure is unique when the infinite Blaschke product $B(z) = \prod_{n=1}^{\infty} \zeta_n(z)$ diverges to zero for

$z \in \mathbb{D}$, i.e., when $\sum_{n=1}^{\infty}(1-|\alpha_n|) = \infty$. This condition means that the sequence $(\alpha_n)_{n\geq 1}$ can not approach to $\mathbb{T}$ very quickly.

Notice that, given a measure μ on $\mathbb{T}$ and a sequence $(\alpha_n)_{n\geq 1}$ in $\mathbb{D}$, the parameters $(a_n)_{n\geq 1}$ are uniquely defined. To see this, suppose that $(\hat{\phi}_n)_{n\geq 0}$ is another sequence of ORF satisfying a recurrence like (10), but with parameters $(\hat{a}_n)_{n\geq 1}$ instead of $(a_n)_{n\geq 1}$. Then, $\hat{\phi}_n = \epsilon_n \phi_n$ with $\epsilon_n \in \mathbb{T}$ and $\epsilon_0 = 1$. Hence, comparing the recurrences for $(\phi_n)_{n\geq 0}$ and $(\hat{\phi}_n)_{n\geq 0}$ gives

$$\frac{1}{\sqrt{1-|a_n|^2}} \begin{pmatrix} \epsilon_n & 0 \\ 0 & \overline{\epsilon}_n \end{pmatrix} \begin{pmatrix} 1 & a_n \\ \overline{a}_n & 1 \end{pmatrix} \begin{pmatrix} \overline{\epsilon}_{n-1} & 0 \\ 0 & \epsilon_{n-1} \end{pmatrix} = \frac{1}{\sqrt{1-|\hat{a}_n|^2}} \begin{pmatrix} 1 & \hat{a}_n \\ \overline{\hat{a}}_n & 1 \end{pmatrix}.$$

Taking determinants in both sides of the above equality we obtain $|\hat{a}_n| = |a_n|$. In consequence $\epsilon_n = \epsilon_{n-1}$ for $n \geq 1$, which yields $\epsilon_n = \epsilon_0 = 1$. Therefore, $\hat{a}_n = a_n$ and $\hat{\phi}_n = \phi_n$.

The above results show that any sequence $\boldsymbol{\alpha} = (\alpha_n)_{n\geq 1}$ in $\mathbb{D}$ defines a surjective application

$$\begin{aligned} \mathcal{S}_{\boldsymbol{\alpha}} \colon \mathfrak{P} &\longrightarrow \mathbb{D}^{\infty} \\ \mu &\longrightarrow \boldsymbol{a} = (a_n)_{n\geq 1} \end{aligned}$$

between the set $\mathfrak{P}$ of probability measures on $\mathbb{T}$ and the set $\mathbb{D}^{\infty}$ of sequences in $\mathbb{D}$. Furthermore, $\mathcal{S}_{\boldsymbol{\alpha}}$ is a bijection when $\sum_{n=1}^{\infty}(1-|\alpha_n|) = \infty$. The study of the application $\mathcal{S}_{\boldsymbol{\alpha}}$ is one of the main interests to find a matrix representation of the multiplication operator T_μ with a simple dependence on the parameters $\boldsymbol{\alpha} = (\alpha_n)_{n\geq 1}$ and $\boldsymbol{a} = (a_n)_{n\geq 1}$. Indeed, in the polynomial case, corresponding to $\boldsymbol{\alpha} = 0$, the five-diagonal representation $\mathcal{C} = \mathcal{C}(\boldsymbol{a})$ of T_μ given in (6) has revealed to be a powerful tool in the study of $\mathcal{S}_0$.

To find such a matrix representation, it is convenient to write recurrence (10) in a different way. For any $\alpha \in \mathbb{D}$ we can define the positive number

$$\eta_\alpha = \varpi_\alpha(\alpha)^{1/2} = \sqrt{1-|\alpha|^2}.$$

Denoting $\eta_n = \eta_{\alpha_n}$ and introducing the parameters

$$\rho_n = \sqrt{1-|a_n|^2}, \qquad \rho_n^+ = \frac{\eta_{n-1}}{\eta_n}\,\rho_n, \qquad \rho_n^- = \frac{\eta_n}{\eta_{n-1}}\,\rho_n, \tag{11}$$

(10) yields

$$\left\{ \begin{aligned} &\varpi_{n-1}^* \phi_{n-1} = \rho_n^+ \varpi_n \phi_n - a_n \varpi_{n-1} \phi_{n-1}^*, \\ &\varpi_n \phi_n^* = \overline{a}_n \varpi_n \phi_n + \rho_n^- \varpi_{n-1} \phi_{n-1}^*, \end{aligned} \right. \qquad n \geq 1. \tag{12}$$

This way of writing (10) will be useful later.

3 Operator Möbius transformations

As we will see, the operator version of the scalar Möbius transformations ζ_α appears in a natural way in the spectral theory of ORF on the unit circle. Analogously to the scalar case, such operator Möbius transformations are a particular case of the linear fractional transformations with operator coefficients introduced by M. G. Krein in [23, 24] for the study of spaces with an indefinite metric. A detailed study of these operator Möbius transformations in the general context of linear fractional transformations can be found, for instance, in the original paper of M. G. Krein and Yu L. Šmuljan [25] or in the most recent survey of T. Ya Azizov and I. S. Iokhvidov [4] and the references therein. We will introduce the operator Möbius transformations summarizing the main properties of interest for us.

Before doing this, we will fix some notations and conventions for linear operators. In what follows $(H, (\cdot,\cdot))$ means a separable Hilbert space. Given a linear operator T on H, $T^\dagger$ denotes its adjoint, $\sigma(T)$ its spectrum and $\sigma_p(T)$ its point spectrum. As usual, we omit the identity operator $\mathbf{1}$ on H so we use the same symbol z for the complex number $z \in \mathbb{C}$ and for the operator $z\mathbf{1}$, the meaning being clear from the context in any case. In general, we will deal with the Banach space $(\mathbb{B}_H, \|\cdot\|)$ of everywhere defined bounded linear operators on H.

In particular, $\mathbb{B}_{\mathbb{C}^n}$ and $\mathbb{B}_{\ell^2}$ can be identified with the sets of $n \times n$ complex matrices and infinite bounded complex matrices respectively. In this identification we associate any bounded square matrix M with the operator $x \to Mx$, where x is a column vector of $\mathbb{C}^n$ or ℓ^2. However, we could also consider the operator $x \to xM$, where x is a row vector of $\mathbb{C}^n$ or ℓ^2. Both operators have the same spectrum, although their eigenvalues can be different in the case of ℓ^2. Nevertheless, we will normally work with normal or finite-dimensional matrices, for which the eigenvalues are the same in both situations. However, even in these cases, the eigenvectors are in general different. So, we will distinguish between right eigenvectors (or just eigenvectors) for $x \to Mx$ and left eigenvectors for $x \to xM$. That is, right eigenvectors are the standard ones while left eigenvectors are the transposed of the eigenvectors of M^T (in particular, when M is normal, right eigenvectors are the adjoints of left eigenvectors). In the subsequent discussions, this convention often permits us to avoid the ^{T} superindex, something convenient because many indices appear later.

The operator Möbius transformations on H are linear fractional trans-

formations with operator coefficients that transform bijectively the unit ball $\mathbb{D}_H = \{T \in \mathbb{B}_H : \|T\| < 1\}$ of $\mathbb{B}_H$ onto itself. The role of the complex parameter $\alpha \in \mathbb{D}$ of ζ_α is played by an operator $A \in \mathbb{D}_H$, so that

$$\eta_A = \sqrt{1 - AA^\dagger}$$

defines a positive operator with bounded inverse. Therefore, for any operator T in the closed unit ball $\overline{\mathbb{D}}_H = \{T \in \mathbb{B}_H : \|T\| \leq 1\}$ we can define the operators $\zeta_A(T), \tilde{\zeta}_A(T) \in \mathbb{B}_H$ by

$$\zeta_A(T) = \eta_A \, \varpi_A(T)^{-1} \varpi_A^*(T) \, \eta_{A^\dagger}^{-1}, \qquad \begin{cases} \varpi_A(T) = 1 - TA^\dagger, \\ \varpi_A^*(T) = T - A, \end{cases}$$

$$\tilde{\zeta}_A(T) = \eta_A^{-1} \tilde{\varpi}_A^*(T) \, \tilde{\varpi}_A(T)^{-1} \eta_{A^\dagger}, \qquad \begin{cases} \tilde{\varpi}_A(T) = 1 + A^\dagger T, \\ \tilde{\varpi}_A^*(T) = T + A. \end{cases}$$

As in the scalar case, $\eta_A = \varpi_A(A)^{1/2}$. As we will see, the spectral theory of ORF is related to transformations $\zeta_A, \tilde{\zeta}_A$ with A normal, so that $\eta_{A^\dagger} = \eta_A$ in such a case.

The transformations ζ_A and $\tilde{\zeta}_A$ are the operator analogs of the scalar Möbius transformations ζ_α and $\tilde{\zeta}_\alpha$ respectively. The factors $\eta_A, \eta_{A^\dagger}$ disappear in the scalar case due to the commutativity. Nevertheless, these factors are necessary for these operator transformations to keep similar properties to the scalar ones. Actually, ζ_A and $\tilde{\zeta}_A$ map $\overline{\mathbb{D}}_H$ on $\overline{\mathbb{D}}_H$, as follows from the identities

$$\begin{aligned} &\varpi_A(T) \, \eta_A^{-1} (1 - \zeta_A(T) \, \zeta_A(T)^\dagger) \, \eta_A^{-1} \varpi_A(T)^\dagger = 1 - TT^\dagger, \\ &\tilde{\varpi}_A(T)^\dagger \, \eta_{A^\dagger}^{-1} (1 - \tilde{\zeta}_A(T)^\dagger \, \tilde{\zeta}_A(T)) \, \eta_{A^\dagger}^{-1} \tilde{\varpi}_A(T) = 1 - T^\dagger T. \end{aligned} \tag{13}$$

Besides, for any $S, T \in \overline{\mathbb{D}}_H$, a direct calculation shows that $S = \zeta_A(T)$ iff $T = \tilde{\zeta}_A(S)$, so ζ_A and $\tilde{\zeta}_A$ are mutually inverse transformations that map $\overline{\mathbb{D}}_H$ onto itself. Furthermore, (13) also proves that ζ_A and $\tilde{\zeta}_A$ leave invariant $\mathbb{D}_H$ and $\mathbb{T}_H = \{T \in \mathbb{B}_H : \|T\| = 1\}$, mapping onto itself the set of isometries as well as the set of unitary operators on H. Indeed, as it was proven in [25], up to unitary left and right factors, these operator Möbius transformations are the only linear fractional transformations with operator coefficients mapping bijectively $\mathbb{D}_H$ onto itself.

Using the relation $\eta_A^2 A = A \eta_{A^\dagger}^2$ it is straightforward to verify the identities

$$\zeta_A(T)^\dagger = \zeta_{A^\dagger}(T^\dagger), \qquad \tilde{\zeta}_A(T)^\dagger = \tilde{\zeta}_{A^\dagger}(T^\dagger), \tag{14}$$

which imply that $\tilde{\zeta}_A(T) = \tilde{\zeta}_{A^\dagger}(T^\dagger)^\dagger = \zeta_{-A}(T)$ as in the scalar case. Notice that the equalities $\zeta_A = \tilde{\zeta}_{-A}$ and $\tilde{\zeta}_A = \zeta_{-A}$ provide alternative expressions for ζ_A and $\tilde{\zeta}_A$.

Some formulas for the operator Möbius transformations will be of interest. From the relations $(\eta_A^2)^n A = A(\eta_{A^\dagger}^2)^n$ for $n = 0, 1, 2, \ldots$, and using the functional calculus for self-adjoint operators, we find that

$$\eta_A \, A = A \, \eta_{A^\dagger}.$$

Thus, if we define

$$T_A = \eta_A^{-1} \, T \, \eta_{A^\dagger}$$

for any linear operator T on H, then, for all $T \in \overline{\mathbb{D}}_H$,

$$\zeta_A(T_A) = \varpi_A(T)^{-1} \varpi_A^*(T), \qquad \tilde{\zeta}_A(T) = \tilde{\varpi}_A^*(T_A) \, \tilde{\varpi}_A(T_A)^{-1}. \tag{15}$$

This, together with the immediate identity

$$\varpi_A^*(T) - \varpi_A(T) \, S = T \, \tilde{\varpi}_A(S) - \tilde{\varpi}_A^*(S), \tag{16}$$

yields

$$\varpi_A(T) \, (\zeta_A(T_A) - S_A) = (T - \tilde{\zeta}_A(S)) \, \tilde{\varpi}_A(S_A) \tag{17}$$

for all $T, S \in \overline{\mathbb{D}}_H$. Substituting S by $\zeta_A(S)$ in (17) gives

$$T - S = \varpi_A(T) \, \eta_A^{-1} (\zeta_A(T) - \zeta_A(S)) \, \eta_{A^\dagger}^{-1} \, \tilde{\varpi}_{-A}(S), \tag{18}$$

where we have used that

$$\tilde{\varpi}_A(\zeta_A(S_A)) = \tilde{\varpi}_A(\tilde{\zeta}_{-A}(S_A)) = \tilde{\varpi}_{-A}(S)^{-1} \eta_{A^\dagger}^2.$$

If we take $A = \alpha$ and $S = z$ with $\alpha \in \mathbb{D}$ and $z \in \overline{\mathbb{D}}$, (18) becomes

$$z - T = \frac{\varpi_\alpha(z)}{\varpi_\alpha(\alpha)} \, (\zeta_\alpha(z) - \zeta_\alpha(T)) \, \varpi_\alpha(T). \tag{19}$$

In particular, choosing $T = \lambda$ with $\lambda \in \overline{\mathbb{D}}$,

$$\zeta_\alpha(z) - \zeta_\alpha(\lambda) = \frac{\varpi_\alpha(\alpha)}{\varpi_\alpha(z) \, \varpi_\alpha(\lambda)} \, (z - \lambda). \tag{20}$$

Notice that (19) and (20) actually hold for any $z, \lambda \in \mathbb{C} \setminus \{\hat{\alpha}\}$.

4 ORF and Hessenberg matrices

In this section we will prove that the orthogonality measure of a sequence of ORF, as well as the zeros of the ORF, have a spectral interpretation in terms of Hessenberg matrices. Our first aim is to find the matrix representation of a unitary multiplication operator with respect to a basis of ORF. Before stating the result, let us see which kind of matrix representation we can expect. Let ν be a measure on $\mathbb{T}$ and $(\varphi_n)_{n\geq 0}$ the corresponding OP with positive leading coefficient. Given $\alpha \in \mathbb{D}$, the functions $\phi_n(z) = \varphi_n(\zeta_\alpha(z))$ define a sequence $(\phi_n)_{n\geq 0}$ of ORF with fixed poles at $\hat{\alpha}$. The corresponding orthogonality measure is $\mu = \nu_\alpha$, where $\nu_\alpha(\Delta) = \nu(\zeta_\alpha(\Delta))$ for any Borel subset Δ of $\mathbb{T}$. It is straightforward to see that recurrence (1) for $(\varphi_n)_{n\geq 0}$ is rewritten in terms of $(\phi_n)_{n\geq 0}$ as recurrence (10) with the same parameters $\boldsymbol{a} = (a_n)_{n\geq 1}$, i.e., $\boldsymbol{a} = \mathcal{S}_0(\nu) = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. The matrix representation of the isometric operator $T_\nu \restriction \mathcal{P}$ with respect to $(\varphi_n)_{n\geq 0}$ is a Hessenberg matrix $\mathcal{H} = \mathcal{H}(\boldsymbol{a})$ with the form (3). Therefore, the matrix of $T_\mu \restriction \mathcal{L}$ with respect to the ORF $(\phi_n)_{n\geq 0}$ is

$$(\langle \phi_i(z), z\phi_j(z)\rangle_\mu)_{i,j=0}^\infty = (\langle \varphi_i(z), \tilde{\zeta}_\alpha(z)\varphi_j(z)\rangle_\nu)_{i,j=0}^\infty = \tilde{\zeta}_\alpha(\mathcal{H}).$$

The following theorem is a natural generalization of this particular situation.

Theorem 4.1. *Let $\boldsymbol{\alpha} = (\alpha_n)_{n\geq 1}$ be compactly included in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $\boldsymbol{a} = (a_n)_{n\geq 1} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. Then, $\mathcal{L}$ is T_μ-invariant and the matrix of the isometric operator $T_\mu \restriction \mathcal{L}$ with respect to the corresponding ORF $(\phi_n)_{n\geq 0}$ is $\mathcal{V} = \tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$, where $\mathcal{H} = \mathcal{H}(\boldsymbol{a})$ is given in (3) and*

$$\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha}) = \begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \alpha_2 & \\ & & & \ddots \end{pmatrix}.$$

The isometric matrix $\mathcal{V}$ represents the full operator T_μ iff any of the following equivalent conditions is fulfilled:

$$\mathcal{L} = L^2_\mu \Leftrightarrow \mathcal{P} = L^2_\mu \Leftrightarrow \log \mu' \notin L^1_m \Leftrightarrow \boldsymbol{a} \notin \ell^2 \Leftrightarrow \mathcal{V} \text{ is unitary.}$$

Proof. $\|\mathcal{A}\| < 1$ because $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$, thus $\tilde{\zeta}_{\mathcal{A}}$ maps onto theirselves the sets of infinite isometric and unitary matrices. Therefore,

taking into account that $\mathcal{H}$ is isometric, $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ is a well defined isometric matrix too.

The starting point to prove the theorem is recurrence (10) written as (12). The second relation in (12) yields

$$\varpi_n \phi_n^* = \overline{a}_n \varpi_n \phi_n + \sum_{k=0}^{n-1} \rho_n^- \rho_{n-1}^- \cdots \rho_{k+1}^- \overline{a}_k \varpi_k \phi_k, \qquad n \geq 1, \tag{21}$$

where we set $a_0 = 1$. This identity, together with the first relation in (12), gives

$$\begin{gathered}\varpi_n^* \phi_n = \sum_{k=0}^{\infty} \hat{h}_{k,n} \varpi_k \phi_k, \\ \hat{h}_{k,n} = \begin{cases} -a_{n+1} \rho_n^- \rho_{n-1}^- \cdots \rho_{k+1}^- \overline{a}_k & \text{if } k < n, \\ -a_{n+1} \overline{a}_n & \text{if } k = n, \\ \rho_{n+1}^+ & \text{if } k = n+1, \\ 0 & \text{if } k > n+1. \end{cases}\end{gathered} \tag{22}$$

If we define the matrix $\hat{\mathcal{H}} = (\hat{h}_{i,j})$, equality (22) can be written as

$$\begin{pmatrix} \phi_0(z) & \phi_1(z) & \cdots \end{pmatrix} \left(\varpi_{\mathcal{A}}^*(z) - \varpi_{\mathcal{A}}(z) \hat{\mathcal{H}} \right) = 0. \tag{23}$$

Using (11) we find that the Hessenberg matrix

$$\hat{\mathcal{H}} = \begin{pmatrix} -a_1 & -\rho_1^- a_2 & -\rho_1^- \rho_2^- a_3 & -\rho_1^- \rho_2^- \rho_3^- a_4 & \cdots \\ \rho_1^+ & -\overline{a}_1 a_2 & -\overline{a}_1 \rho_2^- a_3 & -\overline{a}_1 \rho_2^- \rho_3^- a_4 & \cdots \\ 0 & \rho_2^+ & -\overline{a}_2 a_3 & -\overline{a}_2 \rho_3^- a_4 & \cdots \\ 0 & 0 & \rho_3^+ & -\overline{a}_3 a_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{24}$$

can be related to the isometric Hessenberg matrix $\mathcal{H}$ given in (3) by

$$\hat{\mathcal{H}} = \eta_{\mathcal{A}}^{-1} \mathcal{H} \, \eta_{\mathcal{A}} = \mathcal{H}_{\mathcal{A}}, \tag{25}$$

where we have used that $\eta_{\mathcal{A}^\dagger} = \eta_{\mathcal{A}}$ since $\mathcal{A}$ is diagonal, so normal. From this relation, (15) and (16) we see that (23) is equivalent to

$$\begin{pmatrix} \phi_0(z) & \phi_1(z) & \cdots \end{pmatrix} \left(z - \tilde{\zeta}_{\mathcal{A}}(\mathcal{H}) \right) = 0. \tag{26}$$

This equality implies that $\mathcal{L}$ is invariant under T_μ, so the restriction $T_\mu \restriction \mathcal{L}$ is well defined and $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ is its matrix representation with respect to $(\phi_n)_{n \geq 0}$.

$T_\mu \restriction \mathcal{L}$ is an isometry because it is the restriction of a unitary operator, which agrees with the fact that $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ is isometric. Also, $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ and $\mathcal{H}$ are unitary at the same time, that is, when $\boldsymbol{a} \notin \ell^2$. Besides, $T_\mu \restriction \mathcal{L}$ is unitary iff $T_\mu \mathcal{L} = \mathcal{L}$. This implies that $T_\mu^n \mathcal{L} = \mathcal{L}$ for any $n \in \mathbb{Z}$, so $\{z^n\}_{n\in\mathbb{Z}} \subset \mathcal{L}$. Hence $\mathcal{L} = L^2_\mu$ because $\operatorname{span}\{z^n\}_{n\in\mathbb{Z}}$ is dense in L^2_μ. Conversely, if $\mathcal{L} = L^2_\mu$, then $T_\mu \restriction \mathcal{L} = T_\mu$ is unitary. Therefore, $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ is unitary iff the ORF $(\phi_n)_{n\geq 0}$ are a basis of L^2_μ, i.e., iff $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ represents the full operator T_μ. Finally, it is known that the condition $\sum_{n=1}^\infty (1-|\alpha_n|) = \infty$, which is satisfied for $\boldsymbol{\alpha}$ compactly included in $\mathbb{D}$, ensures that $\mathcal{L} = \mathcal{P}$ (see [10, Theorem 7.2.2]) and so it implies the equivalence between $\mathcal{L} = L^2_\mu$, $\mathcal{P} = L^2_\mu$ and $\log \mu' \notin L^1_m$ (see [10, Corollary 7.2.4]).

□

Given a measure μ on $\mathbb{T}$, the parameters $\boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$ corresponding to the ORF $(\phi_n)_{n\geq 0}$ associated with $\boldsymbol{\alpha}$ are in general different from the parameters $\boldsymbol{a}^{(0)} = \mathcal{S}_0(\mu)$ related to the OP $(\varphi_n)_{n\geq 0}$. For instance, if $\alpha_n = \alpha$ for all n, the comments at the beginning of this section show that $\mathcal{S}_{\boldsymbol{\alpha}}(\mu_\alpha) = \mathcal{S}_0(\mu)$. Taking into account that $\mathcal{S}_{\boldsymbol{\alpha}}$ is a bijection for a constant sequence $\boldsymbol{\alpha}$, we conclude that $\mathcal{S}_{\boldsymbol{\alpha}}(\mu) \neq \mathcal{S}_0(\mu)$. Therefore, the equivalence $\mathcal{P} = L^2_\mu \Leftrightarrow \boldsymbol{a} \notin \ell^2$ is not trivial in the general case since the known result in the polynomial situation is $\mathcal{P} = L^2_\mu \Leftrightarrow \boldsymbol{a}^{(0)} \notin \ell^2$.

Contrary to the polynomial case, the unitary matrix $\mathcal{V}$ of the multiplication operator with respect the ORF basis is not a Hessenberg matrix in general, but $\zeta_{\mathcal{A}}(\mathcal{V})$ is a Hessenberg matrix, where $\zeta_{\mathcal{A}}$ is an operator Möbius transformation constructed using all the poles of the related ORF. In the polynomial case $\mathcal{A} = 0$, thus $\zeta_{\mathcal{A}}(\mathcal{V}) = \mathcal{V}$ and $\mathcal{V}$ becomes a Hessenberg matrix.

As a consequence of Theorem 4.1 and the spectral properties of the unitary multiplication operator, we have the following spectral interpretation of the support of the orthogonality measure for ORF.

Theorem 4.2. *Let $\boldsymbol{\alpha}$ be a sequence compactly included in $\mathbb{D}$, μ a measure on $\mathbb{T}$ such that $\log \mu' \notin L^1_m$ and $(\phi_n)_{n\geq 0}$ the corresponding ORF. If*

$$\mathcal{V} = \tilde{\zeta}_{\mathcal{A}}(\mathcal{H}), \qquad \mathcal{A} = \mathcal{A}(\boldsymbol{\alpha}), \qquad \mathcal{H} = \mathcal{H}(\boldsymbol{a}), \qquad \boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu),$$

and $\mathcal{E}$ is the spectral measure of $\mathcal{V}$, then $\mu = \mathcal{E}_{1,1}$. Besides, $\operatorname{supp}\mu = \sigma(\mathcal{V})$ and the mass points of μ are the eigenvalues of $\mathcal{V}$, which have geometric multiplicity 1. λ is a mass point iff $(\phi_n(\lambda))_{n\geq 0} \in \ell^2$. Given a mass point

λ, *the corresponding eigenvectors of* $\mathcal{V}$ *are spanned by* $(\,\phi_0(\lambda) \quad \phi_1(\lambda) \quad \cdots)^\dagger$ *and* $\mu(\{\lambda\}) = (\sum_{n=0}^{\infty} |\phi_n(\lambda)|^2)^{-1}$.

Proof. Under the hypothesis of the theorem, $\mathcal{V}$ is the matrix representation of the full operator T_μ with respect to $(\phi_n)_{n\geq 0}$. Hence, if E is the spectral measure of T_μ, then $\mu(\cdot) = \langle \phi_0, E(\cdot)\phi_0\rangle_\mu = \mathcal{E}_{1,1}(\cdot)$. Also, supp$\mu = \sigma(T_\mu) = \sigma(\mathcal{V})$ and the mass points of μ are the eigenvalues of T_μ, that is, the eigenvalues of $\mathcal{V}$, which have therefore geometric multiplicity 1. If λ is a mass point, we know that the characteristic function $\mathcal{X}_{\{\lambda\}}$ of $\{\lambda\}$ is a related eigenvector of T_μ, so, $\langle \phi_n, \mathcal{X}_{\{\lambda\}}\rangle_\mu = \mu(\{\lambda\})\,\overline{\phi_n(\lambda)}$ is the $(n+1)$-th component of a corresponding eigenvector of $\mathcal{V}$. This implies that $(\phi_n(\lambda))_{n\geq 0} \in \ell^2$. Conversely, if λ is an arbitrary complex number such that $(\phi_n(\lambda))_{n\geq 0} \in \ell^2$, relation (26) shows that $(\,\phi_0(\lambda) \quad \phi_1(\lambda) \quad \cdots)$ is a left eigenvector of $\mathcal{V}$ with eigenvalue λ. Due to the unitarity of $\mathcal{V}$, $\lambda \in \mathbb{T}$ and the above statement is equivalent to saying that $(\,\phi_0(\lambda) \quad \phi_1(\lambda) \quad \cdots)^\dagger$ is a (right) eigenvector of $\mathcal{V}$ with eigenvalue λ. Therefore, λ is a mass point of μ. Also, the identity

$$\mu(\{\lambda\}) = \langle \mathcal{X}_{\{\lambda\}}, \mathcal{X}_{\{\lambda\}}\rangle_\mu = \sum_{n=0}^{\infty} \langle \mathcal{X}_{\{\lambda\}}, \phi_n\rangle_\mu \langle \phi_n, \mathcal{X}_{\{\lambda\}}\rangle_\mu = \sum_{n=0}^{\infty} \mu(\{\lambda\})^2 |\phi_n(\lambda)|^2$$

proves that $\mu(\{\lambda\}) = (\sum_{n=0}^{\infty} |\phi_n(\lambda)|^2)^{-1}$.

□

The fact that the representation $\mathcal{V}$ is not a Hessenberg matrix, but a Möbius transformation of a Hessenberg matrix, makes the rational case more complicated than the polynomial one. However, the Hessenberg structure can be kept if we formulate the spectral results in terms of pairs of operators.

Remember that, given a Hilbert space H and two operators $T, S \in \mathbb{B}_H$, the spectrum and point spectrum of the pair (T, S) are respectively the sets

$$\sigma(T,S) = \{\lambda \in \overline{\mathbb{C}} : T - \lambda S \text{ has no inverse in } \mathbb{B}_H\},$$
$$\sigma_p(T,S) = \{\lambda \in \overline{\mathbb{C}} : T - \lambda S \text{ is not injective}\}.$$

In the finite-dimensional case both sets coincide. The elements of $\sigma_p(T,S)$ are called eigenvalues of the pair, and the eigenvectors of (T,S) corresponding to an eigenvalue λ are the elements $x \in H \setminus \{0\}$ such that $(T - \lambda S)x = 0$. In these definitions it is assumed that, if $\lambda = \infty$, $T - \lambda S$ must be substituted by S.

With the above terminology, the isometric matrix $\mathcal{V}$ and the Hessenberg pair $(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}))$ have the same spectrum and eigenvalues because $\tilde{\varpi}_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}})^{\pm 1} \in \mathbb{B}_{\ell^2}$ when $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$. So, Theorem 4.2 can be obviously rewritten substituting $\mathcal{V}$ by the pair $(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}))$. Notice that, given an eigenvalue λ, $(\,\phi_0(\lambda) \quad \phi_1(\lambda) \quad \cdots\,)$ is a left eigenvector of the pair, i.e.,

$$(\,\phi_0(\lambda) \quad \phi_1(\lambda) \quad \cdots\,)\,(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}) - \lambda\,\tilde{\varpi}_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}})) = 0.$$

Moreover, $\mathcal{H}_{\mathcal{A}} = \eta_{\mathcal{A}}^{-1}\mathcal{H}\,\eta_{\mathcal{A}}$ with $\eta_{\mathcal{A}}^{\pm 1} \in \mathbb{B}_{\ell^2}$ due to the restrictions on $\boldsymbol{\alpha}$. Therefore, Theorem 4.2 also holds substituting $\mathcal{V}$ by the Hessenberg pair $(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{H}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{H}))$, but the left eigenvectors with eigenvalue λ are spanned by $(\,\phi_0(\lambda) \quad \phi_1(\lambda) \quad \cdots\,)\,\eta_{\mathcal{A}}^{-1}$.

4.1 Zeros of ORF and Hessenberg matrices

Let μ be a measure on $\mathbb{T}$ and $\boldsymbol{a} = \mathcal{S}_0(\mu)$. As we pointed out in Section 2, the characteristic polynomial of the orthogonal truncation of T_μ on $\mathcal{P}_n = \operatorname{span}\{1, z, \ldots, z^{n-1}\}$ is a multiple of the n-th OP related to μ. From this result, the relation between the zeros of the n-th OP and the eigenvalues of the principal submatrices of $\mathcal{H}(\boldsymbol{a})$ follows.

To obtain a similar result for the ORF associated with a sequence $\boldsymbol{\alpha}$ we have to consider the operator multiplication by ζ_n in L^2_μ, i.e.,

$$\zeta_n(T_\mu)\colon \begin{array}{c} L^2_\mu \to L^2_\mu \\ f \longrightarrow \zeta_n f \end{array}$$

and the orthogonal truncation of $\zeta_n(T_\mu)$ on $\mathcal{L}_n$. This orthogonal truncation is defined by $\zeta_n(T_\mu)^{(\mathcal{L}_n)} = L_n \zeta_n(T_\mu) \restriction \mathcal{L}_n$, where the operator $L_n\colon L^2_\mu \to L^2_\mu$ is the orthogonal projection on $\mathcal{L}_n$. The following theorem is the starting point to identify the zeros of the ORF as the eigenvalues of some finite matrices related to $\mathcal{A}(\boldsymbol{\alpha})$ and $\mathcal{H}(\boldsymbol{a})$.

We remind that the n-th ORF ϕ_n has the form

$$\phi_n = \frac{p_n}{\pi_n}, \qquad p_n \in \mathcal{P}_{n+1} \setminus \mathcal{P}_n, \qquad \pi_n = \varpi_1 \cdots \varpi_n,$$

and the zeros of ϕ_n, which are the zeros of the polynomial p_n, lie on $\mathbb{D}$ (see [10, Corollary 3.2.2]).

Theorem 4.3. *Let $\boldsymbol{\alpha}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $\phi_n = p_n/\pi_n$ the related n-th ORF. Then:*

1. *If Z_n is the set of zeros of ϕ_n, $\zeta_n(Z_n)$ is the set of eigenvalues of $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$ and these eigenvalues have geometric multiplicity 1.*
2. *If $p_n(z) \propto \prod_{k=1}^n (z-\lambda_k)$, the characteristic polynomial of $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$ is*
$$\prod_{k=1}^n (z - \zeta_n(\lambda_k)).$$

Proof. $f \in \mathcal{L}_n \setminus \{0\}$ is an eigenvector of $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$ with eigenvalue w iff $(L_n\zeta_n - w)f = 0$, that is, $L_n(\zeta_n - w)f = 0$. This is equivalent to state that $(\zeta_n - w)f \in \mathcal{L}_n^{\perp\mathcal{L}_{n+1}} = \operatorname{span}\{\phi_n\}$, or, in other words, $f \propto \phi_n(\zeta_n - w)^{-1}$. Writing $w = \zeta_n(\lambda)$ and using (20) we find that this condition can be expressed as $f(z) \propto p_n(z)(z-\lambda)^{-1}/\pi_{n-1}(z)$ with $\lambda \in Z_n$. This proves item 1.

Item 2 is equivalent to assert that the algebraic multiplicity m_w of any eigenvalue $w = \zeta_n(\lambda)$ of $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$ is equal to the multiplicity of λ as a root of p_n. Since the geometric multiplicity of w is 1, $m_w \geq k$ iff there exists $f \in \mathcal{L}_n$ such that $(L_n\zeta_n - w)^k f = 0$ and $(L_n\zeta_n - w)^{k-1} f \neq 0$. Analogously to the previous discussion, we find that these two conditions are equivalent to $f \in \operatorname{span}\{\phi_n(\zeta_n - w)^{-j}\}_{j=1}^k \setminus \operatorname{span}\{\phi_n(\zeta_n - w)^{-j}\}_{j=1}^{k-1}$, i.e., to $f = p/\pi_{n-1}$ with $p(z) \in \operatorname{span}\{\varpi_n^{j-1}(z)p_n(z)(z-\lambda)^{-j}\}_{j=1}^k \setminus \operatorname{span}\{\varpi_n^{j-1}(z)p_n(z)(z-\lambda)^{-j}\}_{j=1}^{k-1}$, as can be seen using (20) again. Hence, by induction on k we find that $m_w \geq k$ implies that the multiplicity of λ as a root of p_n is not less than k. Conversely, if the multiplicity of λ as a root of p_n is greater than or equal to k, $f(z) = \phi_n(z)(\zeta_n(z) - w)^{-k} \propto \varpi_n^{k-1}(z)p_n(z)(z-\lambda)^{-k}/\pi_{n-1}(z) \in \mathcal{L}_n$ and the above results ensure that $(L_n\zeta_n - w)^k f = 0$ and $(L_n\zeta_n - w)^{k-1} f \neq 0$, so $m_w \geq k$.

□

The next step is to obtain a matrix representation of the orthogonal truncation $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$, so that we can give a matrix version of the above theorem. In the following results the subscript $_n$ on a matrix means the principal submatrix of order n of such a matrix. This notation will be used throughout the rest of the paper.

Theorem 4.4. *Let $\boldsymbol{\alpha}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $(\phi_n)_{n\geq 0}$ the related ORF. If $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$ and $\mathcal{H} = \mathcal{H}(\boldsymbol{a})$ with $\boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$, the matrix of $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$ with respect to $(\phi_k)_{k=0}^{n-1}$ is $\zeta_n(\mathcal{V}^{(n)})$, where*

$$\mathcal{V}^{(n)} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{H}_n), \qquad \|\mathcal{V}^{(n)}\| = 1.$$

Proof. $\|\mathcal{A}_n\| < 1$, thus $\tilde{\zeta}_{\mathcal{A}_n}$ maps $\mathbb{T}_{\mathbb{C}^n}$ onto itself. Also, from (5) we obtain the factorization

$$\mathcal{H}_n = \begin{pmatrix} \Theta_1 & \\ & I_{n-2} \end{pmatrix} \begin{pmatrix} I_1 & & \\ & \Theta_2 & \\ & & I_{n-3} \end{pmatrix} \cdots \begin{pmatrix} I_{n-2} & \\ & \Theta_{n-1} \end{pmatrix} \begin{pmatrix} I_{n-1} & \\ & -a_n \end{pmatrix},$$

all the factors being unitary except the last one which has norm 1, thus $\|\mathcal{H}_n\| = 1$. Hence, $\mathcal{V}^{(n)} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{H}_n)$ is well defined and $\|\mathcal{V}^{(n)}\| = 1$. A similar reason shows that $\zeta_n(\mathcal{V}^{(n)})$ is a well defined matrix with norm 1.

To prove the theorem, let us write the first n equations of (23) as

$$\begin{aligned} &(\,\phi_0(z) \;\; \cdots \;\; \phi_{n-1}(z)\,)\left(\varpi^*_{\mathcal{A}_n}(z) - \varpi_{\mathcal{A}_n}(z)\,\hat{\mathcal{H}}_n\right) = b_n \varpi_n(z)\phi_n(z), \\ &b_n = \rho_n^+ \,(\,0 \quad 0 \quad \cdots \quad 0 \quad 1\,) \in \mathbb{C}^n. \end{aligned} \tag{27}$$

Then, identities (15), (16) and the equality $\hat{\mathcal{H}}_n = \eta_{\mathcal{A}_n}^{-1} \mathcal{H}_n \, \eta_{\mathcal{A}_n} = (\mathcal{H}_n)_{\mathcal{A}_n}$ obtained from (25) transform (27) into

$$(\,\phi_0(z) \quad \cdots \quad \phi_{n-1}(z)\,)\left(z - \mathcal{V}^{(n)}\right) = c_n \varpi_n(z)\phi_n(z), \qquad c_n \in \mathbb{C}^n. \tag{28}$$

Using (19) we get

$$(\,\phi_0(z) \quad \cdots \quad \phi_{n-1}(z)\,)\left(\zeta_n(z) - \zeta_n(\mathcal{V}^{(n)})\right) = d_n \phi_n(z), \qquad d_n \in \mathbb{C}^n.$$

Hence, taking into account that

$$L_n \phi_k = \begin{cases} \phi_k & \text{if } k < n, \\ 0 & \text{if } k \geq n, \end{cases}$$

we finally obtain

$$(\,L_n\zeta_n\phi_0 \quad \cdots \quad L_n\zeta_n\phi_{n-1}\,) = (\,\phi_0 \quad \cdots \quad \phi_{n-1}\,)\,\zeta_n(\mathcal{V}^{(n)}),$$

which proves that $\zeta_n(\mathcal{V}^{(n)})$ is the matrix of $L_n\zeta_n(T_\mu) \restriction \mathcal{L}_n$ with respect to $(\phi_k)_{k=0}^{n-1}$.

$\square$

Theorems 4.3 and 4.4 provide a spectral interpretation of the zeros of ORF in terms of operator Möbius transformations of Hessenberg matrices.

Theorem 4.5. *Let $\boldsymbol{\alpha}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geq 0}$ the corresponding ORF, $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$ and $\mathcal{H} = \mathcal{H}(\boldsymbol{a})$ with $\boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. If $\mathcal{V}^{(n)} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{H}_n)$, then:*

1. *The zeros of ϕ_n are the eigenvalues of $\mathcal{V}^{(n)}$, which have geometric multiplicity 1. If λ is a zero of ϕ_n, the related left eigenvectors of $\mathcal{V}^{(n)}$ are spanned by $\begin{pmatrix} \phi_0(\lambda) & \cdots & \phi_{n-1}(\lambda) \end{pmatrix}$.*

2. *$\phi_n = \dfrac{p_n}{\pi_n}$ with p_n proportional to the characteristic polynomial of $\mathcal{V}^{(n)}$.*

Proof. From Theorems 4.3 and 4.4, the eigenvalues of $\zeta_n(\mathcal{V}^{(n)})$ have geometric multiplicity 1 and $\sigma(\zeta_n(\mathcal{V}^{(n)})) = \zeta_n(Z_n)$, where Z_n are the zeros of ϕ_n. Also, the characteristic polynomial of $\zeta_n(\mathcal{V}^{(n)})$ is $\prod_{k=1}^n (z - \zeta_n(\lambda_k))$, where $p_n(z) = \prod_{k=1}^n (z - \lambda_k)$. $\sigma(\zeta_n(\mathcal{V}^{(n)})) = \zeta_n(\sigma(\mathcal{V}^{(n)}))$, so, bearing in mind that ζ_n is bijective, $\sigma(\mathcal{V}^{(n)}) = Z_n$. Furthermore, given an eigenvalue λ of $\mathcal{V}^{(n)}$, the corresponding eigenvalue $\zeta_n(\lambda)$ of $\zeta_n(\mathcal{V}^{(n)})$ has the same geometric and algebraic multiplicity. Therefore, $\prod_{k=1}^n (z - \lambda_k)$ is the characteristic polynomial of $\mathcal{V}^{(n)}$. Finally, if λ is a zero of ϕ_n, (28) shows that $\begin{pmatrix} \phi_0(\lambda) & \cdots & \phi_{n-1}(\lambda) \end{pmatrix}$ is a left eigenvector of $\mathcal{V}^{(n)}$ with eigenvalue λ.

$\square$

From (15) and (16) we know that

$$\varpi^*_{\mathcal{A}_n}(z) - \varpi_{\mathcal{A}_n}(z)\,\hat{\mathcal{H}}_n = z\,\tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{H}}_n) - \tilde{\varpi}^*_{\mathcal{A}_n}(\hat{\mathcal{H}}_n) = (z - \mathcal{V}^{(n)})\,\tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{H}}_n).$$

This gives other alternatives to express p_n as a determinant, like

$$p_n(z) \propto \det\left(\varpi^*_{\mathcal{A}_n}(z) - \varpi_{\mathcal{A}_n}(z)\,\hat{\mathcal{H}}_n\right) = \det\left(z\,\tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{H}}_n) - \tilde{\varpi}^*_{\mathcal{A}_n}(\hat{\mathcal{H}}_n)\right).$$

The interest of the above expressions is that they show that p_n can be calculated as a determinant of a Hessenberg matrix. Furthermore, the last expression provides a new spectral interpretation of the zeros of the ORF, related to the concept of the spectrum of a pair of operators. It shows that the zeros of ϕ_n are the eigenvalues of the Hessenberg pair $(\tilde{\varpi}^*_{\mathcal{A}_n}(\hat{\mathcal{H}}_n), \tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{H}}_n))$. Also, according to Theorem 4.5, the left eigenvectors of $(\tilde{\varpi}^*_{\mathcal{A}_n}(\hat{\mathcal{H}}_n), \tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{H}}_n))$ corresponding to an eigenvalue λ are spanned by $\begin{pmatrix} \phi_0(\lambda) & \cdots & \phi_{n-1}(\lambda) \end{pmatrix}$. Taking into account that $\hat{\mathcal{H}}_n = \eta^{-1}_{\mathcal{A}_n} \mathcal{H}_n\, \eta_{\mathcal{A}_n}$, the zeros of ϕ_n can be also understood as the eigenvalues of the Hessenberg pair $(\tilde{\varpi}^*_{\mathcal{A}_n}(\mathcal{H}_n), \tilde{\varpi}_{\mathcal{A}_n}(\mathcal{H}_n))$, the left

eigenvectors with eigenvalue λ being spanned by $(\,\phi_0(\lambda) \quad \cdots \quad \phi_{n-1}(\lambda)\,)\,\eta_{\mathcal{A}_n}^{-1}$. Indeed,

$$p_n(z) \propto \det\left(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\,\mathcal{H}_n\right) = \det\left(z\,\tilde{\varpi}_{\mathcal{A}_n}(\mathcal{H}_n) - \tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{H}_n)\right).$$

Apart from the sequence $(\phi_n)_{n\geq 0}$ of ORF, another remarkable rational functions arise in the theory of ORF. They are the so called para-orthogonal rational functions (PORF), given by

$$Q_n^v = \phi_n + v\phi_n^*, \qquad v \in \mathbb{T}. \tag{29}$$

The PORF are the generalization of the POP to the rational case. Analogously to the POP, the interest of the PORF Q_n^v relies on the fact that, contrary to the ORF ϕ_n, it has n different zeros lying on $\mathbb{T}$ which, thus, play an important role in quadrature formulas and rational moment problems (see [10, Chapters 5 and 10]). These quadrature formulas associate with each PORF Q_n^v a measure μ_n^v supported on its zeros with a mass $(\sum_{k=0}^{n-1}|\phi_k(\lambda)|^2)^{-1}$ at each zero λ. Such quadrature formulas are exact in $\mathcal{L}_{n-1}\mathcal{L}_{n-1*}$.

Due to the exactness of the quadrature formulas, the first n OP $(\phi_k)_{k=0}^{n-1}$ related to μ are also an orthonormal basis of the n-dimensional Hilbert space $L^2_{\mu_n^v}$. The convergence properties of the quadrature formulas imply that, for any sequence $(v_n)_{n\geq 1}$ in $\mathbb{T}$, the sequence $(\mu_n^{v_n})_{n\geq 1}$ of measures $*$-weakly converges to the orthogonality measure μ on $\mathbb{T}$ whenever $\sum_{n=1}^\infty(1-|\alpha_n|) = \infty$.

A spectral interpretation can be also obtained for the zeros of the PORF. To understand this, let us write the PORF in an equivalent way. Using recurrence (10) we find that

$$Q_n^v = (1+\overline{a}_n v)\,e_n \frac{\varpi_{n-1}}{\varpi_n}\,(\zeta_{n-1}\phi_{n-1} + u\phi_{n-1}^*), \qquad u = \tilde{\zeta}_{a_n}(v). \tag{30}$$

Notice that the parameter u goes through the full unit circle as the parameter v does so. (30) shows that, like ϕ_n, Q_n^v is obtained from n steps of recurrence (10), but changing in the n-th step $a_n \in \mathbb{D}$ by $u = \tilde{\zeta}_{a_n}(v) \in \mathbb{T}$. The analogous substitution in $\mathcal{H}_n$ gives

$$\mathcal{H}_n^u = \begin{pmatrix} \Theta_1 & \\ & I_{n-2} \end{pmatrix} \begin{pmatrix} I_1 & & \\ & \Theta_2 & \\ & & I_{n-3} \end{pmatrix} \cdots \begin{pmatrix} I_{n-2} & \\ & \Theta_{n-1} \end{pmatrix} \begin{pmatrix} I_{n-1} & \\ & -u \end{pmatrix},$$

which is obviously a unitary matrix. Therefore, $\tilde{\zeta}_{\mathcal{A}_n}(\mathcal{H}_n^u)$ is unitary too because $\tilde{\zeta}_{\mathcal{A}_n}$ preserves the unitarity.

The following result provides a spectral interpretation of the zeros of the PORF Q_n^v in terms of the unitary Hessenberg matrix $\mathcal{H}_n^u$, as well as a connection of such a matrix with the unitary multiplication operator $T_{\mu_n^v}$. It can be understood as a limit case of Theorems 4.4 and 4.5.

Theorem 4.6. *Let $\boldsymbol{\alpha}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geq 0}$ the corresponding ORF, $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$ and $\mathcal{H} = \mathcal{H}(\boldsymbol{a})$ with $\boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. If $Q_n^v = \phi_n + v\phi_n^*$ is the n-th PORF related to $v \in \mathbb{T}$ and μ_n^v is the associated measure, then:*

1. *The matrix of $T_{\mu_n^v}$ with respect to $(\phi_k)_{k=0}^{n-1}$ is*

$$\mathcal{V}^{(n;u)} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{H}_n^u), \qquad u = \tilde{\zeta}_{a_n}(v).$$

2. *The zeros of Q_n^v are the eigenvalues of $\mathcal{V}^{(n;u)}$. If λ is a zero of Q_n^v, the related eigenvectors of $\mathcal{V}^{(n;u)}$ are spanned by $(\,\phi_0(\lambda) \quad \cdots \quad \phi_{n-1}(\lambda)\,)^\dagger$.*

3. *$Q_n^v = \dfrac{q_n^v}{\pi_n}$ with q_n^v proportional to the characteristic polynomial of $\mathcal{V}^{(n;u)}$.*

Proof. Using (21) in (30) we find that

$$\varpi_{n-1}^*\phi_{n-1} = \sum_{k=0}^{n-1} \hat{h}_{k,n-1}^u \varpi_k\phi_k + \frac{\rho_n^+}{1+\overline{a}_n v}\,\varpi_n Q_n^v,$$

$$\hat{h}_{k,n-1}^u = \begin{cases} -u\rho_{n-1}^-\rho_{n-2}^-\cdots\rho_{k+1}^-\overline{a}_k & \text{if } k < n-1, \\ -u\overline{a}_{n-1} & \text{if } k = n-1, \end{cases} \qquad u = \tilde{\zeta}_{a_n}(v).$$

This relation together with the first $n-1$ equations of (23) lead to the matrix identity

$$(\,\phi_0(z) \ \cdots \ \phi_{n-1}(z)\,)\left(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\,\hat{\mathcal{H}}_n^u\right) = b_n\varpi_n(z)Q_n^v(z),$$
$$b_n = \frac{\rho_n^+}{1+\overline{a}_n v}\,(\,0 \quad 0 \quad \cdots \quad 0 \quad 1\,) \in \mathbb{C}^n.$$

where $\hat{\mathcal{H}}_n^u = \eta_{\mathcal{A}_n}^{-1}\mathcal{H}_n^u\,\eta_{\mathcal{A}_n} = (\mathcal{H}_n^u)_{\mathcal{A}_n}$. So, (15) and (16) give

$$(\,\phi_0(z) \quad \cdots \quad \phi_{n-1}(z)\,)\left(z - \mathcal{V}^{(n;u)}\right) = c_n\varpi_n(z)Q_n^v(z), \qquad c_n \in \mathbb{C}^n. \tag{31}$$

$Q_n^v = 0$ in $L^2_{\mu_n^v}$, thus (31) implies that $\mathcal{V}^{(n;u)}$ is the matrix of $T_{\mu_n^v}$ with respect to $(\phi_k)_{k=0}^{n-1}$. The rest of the statements are a consequence of this one and the

properties of the multiplication operators, similarly to the proof of Theorem 4.2. Alternatively, they can be obtained directly from relation (31), the unitarity of $\mathcal{V}^{(n:u)}$ and the fact that Q_n^v has n different zeros.

$\square$

Analogously to the comments after Theorem 4.5, if $u = \tilde{\zeta}_{a_n}(v)$,

$$q_n^v(z) \propto \det\left(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\,\mathcal{H}_n^u\right) = \det\left(z\,\tilde{\varpi}_{\mathcal{A}_n}(\mathcal{H}_n^u) - \tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{H}_n^u)\right),$$

which gives q_n^v as a determinant of a Hessenberg matrix too. The zeros of Q_n^v are the eigenvalues of the Hessenberg pair $(\tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{H}_n^u), \tilde{\varpi}_{\mathcal{A}_n}(\mathcal{H}_n^u))$, whose left eigenvectors with eigenvalue λ are spanned by $\begin{pmatrix} \phi_0(\lambda) & \cdots & \phi_{n-1}(\lambda) \end{pmatrix} \eta_{\mathcal{A}_n}^{-1}$.

5 ORF and five-diagonal matrices

Apart from the presence of operator Möbius transformations, there are some drawbacks in the spectral theory of ORF previously developed: the appearance of a Hessenberg matrix $\mathcal{H}$ instead of a band one, the complicated dependence of $\mathcal{H} = \mathcal{H}(\boldsymbol{a})$ on the parameters $\boldsymbol{a}$, and the fact that it represents the full multiplication operator T_μ only for certain measures μ on $\mathbb{T}$. We will not be able to avoid the operator Möbius transformations because they are linked to the ORF, but the other problems can be overcome by choosing a different basis of ORF in L^2_μ.

The key idea is to use, instead of the ORF $(\phi_n)_{n\geq 0}$ with poles in $\mathbb{E}$, other ones whose poles are alternatively in $\mathbb{E}$ and $\mathbb{D}$. For this pourpose we define the finite odd and even Blaschke products

$$\begin{aligned} &B_0^o = B_0^e = 1, \\ &B_n^o = \zeta_1\zeta_3\cdots\zeta_{2n-1}, \qquad B_n^e = \zeta_2\zeta_4\cdots\zeta_{2n}, \qquad n \geq 1. \end{aligned}$$

Consider the rational functions $(\chi_n)_{n\geq 0}$ given by

$$\chi_{2n} = B_{n*}^e \phi_{2n}^*, \qquad \chi_{2n+1} = B_{n*}^e \phi_{2n+1}, \qquad n \geq 0. \tag{32}$$

Since $\zeta_{n*} = 1/\zeta_n$, the subspaces $\mathcal{M}_n = \operatorname{span}\{\chi_0, \dots, \chi_{n-1}\}$ are

$$\begin{aligned} \mathcal{M}_{2n} &= B_{n-1*}^e \mathcal{L}_{2n} = \operatorname{span}\{B_{0*}^e, B_1^o, B_{1*}^e, \dots, B_{n-1*}^e, B_n^o\}, \\ \mathcal{M}_{2n+1} &= B_{n*}^e \mathcal{L}_{2n+1} = \operatorname{span}\{B_{0*}^e, B_1^o, B_{1*}^e, \dots, B_n^o, B_{n*}^e\}, \end{aligned}$$

i.e., $\mathcal{M}_{2n}$ and $\mathcal{M}_{2n+1}$ are the sets of rational functions whose poles, counted with multiplicity, lie on $(\hat{\alpha}_1, \alpha_2, \dots, \alpha_{2n-2}, \hat{\alpha}_{2n-1})$ and $(\hat{\alpha}_1, \alpha_2, \dots, \hat{\alpha}_{2n-1}, \alpha_{2n})$ respectively. We will use the notation $\mathcal{M}_\infty = \operatorname{span}\{\chi_n\}_{n\geq 0} = \cup_{n\geq 1}\mathcal{M}_n$ and $\mathcal{M}$ for the closure of $\mathcal{M}_\infty$ in L^2_μ.

The orthonormality conditions $\phi_n \perp \mathcal{L}_n$ and $\langle \phi_n, \phi_n\rangle_\mu = 1$ can be rewritten using ϕ_n^* as $\phi_n^* \perp \zeta_n \mathcal{L}_n$ and $\langle \phi_n^*, \phi_n^*\rangle_\mu = 1$. Hence, the orthonormality of $(\phi_n)_{n\geq 0}$ is equivalent to $\chi_{2n} \perp B^e_{n*}\zeta_{2n}\mathcal{L}_{2n} = \mathcal{M}_{2n}$, $\chi_{2n+1} \perp B^e_{n*}\mathcal{L}_{2n+1} = \mathcal{M}_{2n+1}$ and $\langle \chi_n, \chi_n\rangle_\mu = 1$, i.e., to the orthonormality of $(\chi_n)_{n\geq 0}$. The sequence $(\chi_n)_{n\geq 0}$ is therefore the result of orthonormalizing $(B^e_{0*}, B^o_1, B^e_{1*}, B^o_2, B^e_{2*}, \dots)$ in L^2_μ. Hence, relation (32) establishes a bijection between ORF associated with the sequences $(\alpha_n)_{n\geq 1}$ and $(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \dots)$. We can consider also the ORF associated with the sequence $(\hat{\alpha}_1, \alpha_2, \hat{\alpha}_3, \alpha_4, \dots)$, i.e., the ORF that arise from the orthonormalization of $(B^e_0, B^o_{1*}, B^e_1, B^o_{2*}, B^e_2, \dots)$ in L^2_μ. This ORF are $(\chi_{n*})_{n\geq 0}$, which are related to $(\phi_n)_{n\geq 0}$ by

$$\chi_{2n*} = B^o_{n*}\phi_{2n}, \qquad \chi_{2n+1*} = B^o_{n+1*}\phi^*_{2n+1}, \qquad n \geq 0. \tag{33}$$

As a conclusion, the two possibilities to generate ORF which alternate poles in $\mathbb{E}$ and $\mathbb{D}$ are related between them and, also, to the ORF with poles in $\mathbb{E}$. The last ORF have been extensively studied, thus every known result for them can be easily translated to the first ones. This is an interesting result because the ORF with poles arbitrarily located in $\overline{\mathbb{C}}\setminus\mathbb{T} = \mathbb{E}\cup\mathbb{D}$ are not so well known than the ORF with poles in $\mathbb{E}$. However, surprisingly, we will use the above connection to obtain new results for ORF with poles in $\mathbb{E}$ using ORF with alternating poles in $\mathbb{E}$ and $\mathbb{D}$. The basic idea is that the ORF $(\chi_n)_{n\geq 0}$ provide new matrix tools for the analysis of questions concerning the ORF $(\phi_n)_{n\geq 0}$. The reason for this is the different nature of the recurrence satisfied by $(\chi_n)_{n\geq 0}$, which, as we will see, is a five-term linear recurrence relation. We could think that recurrence (10) for $(\phi_n)_{n\geq 0}$ should be better because it is two-term, but the presence of ϕ_n^* causes a non-linearity which is the origin of the difficulties to connect with linear operator theory. Alternatively, expanding ϕ_n^* in the basis $(\phi_n)_{n\geq 0}$ we find the linear relation (22), but this is a recurrence with a number of terms that increases with n, giving rise to a representation of T_μ related to a Hessenberg instead of a band matrix. On the contrary, the five-term linear recurrence for the ORF $(\chi_n)_{n\geq 0}$ provides a matrix representation of T_μ in terms of five-diagonal matrices, as the following theorem states.

Theorem 5.1. *Let $\boldsymbol{\alpha}$ be a sequence compactly included in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $\boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. Then, the ORF $(\chi_n)_{n\geq 0}$ associated with $(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$ are a basis of L^2_μ and the matrix of T_μ with respect to $(\chi_n)_{n\geq 0}$ is $\mathcal{U} = \tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$, where $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$ and $\mathcal{C} = \mathcal{C}(\boldsymbol{a})$ is given in (6).*

Proof. Since $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$, $\|\mathcal{A}\| < 1$ and, hence, $\tilde{\zeta}_{\mathcal{A}}$ maps unitarity matrices into unitary matrices. Thus $\tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ is a well defined unitary matrix because $\mathcal{C}$ is unitary.

Using (12) and (32) we find that, for $n \geq 1$,

$$
\begin{aligned}
\varpi^*_{2n-1}\chi_{2n-1} &= B^e_{n-1*}(\rho^+_{2n}\varpi_{2n}\phi_{2n} - a_{2n}\varpi_{2n-1}\phi^*_{2n-1}) = \\
&= B^e_{n*}\rho^+_{2n}(\rho^+_{2n+1}\varpi_{2n+1}\phi_{2n+1} - a_{2n+1}\varpi_{2n}\phi^*_{2n}) - \\
&\quad - B^e_{n-1*}a_{2n}(\overline{a}_{2n-1}\varpi_{2n-1}\phi_{2n-1} + \rho^-_{2n-1}\varpi_{2n-2}\phi^*_{2n-2}) = \\
&= \rho^+_{2n}\rho^+_{2n+1}\varpi_{2n+1}\chi_{2n+1} - \rho^+_{2n}a_{2n+1}\varpi_{2n}\chi_{2n} - \\
&\quad - \overline{a}_{2n-1}a_{2n}\varpi_{2n-1}\chi_{2n-1} - \rho^-_{2n-1}a_{2n}\varpi_{2n-2}\chi_{2n-2}, \\
\varpi^*_{2n}\chi_{2n} &= B^e_{n-1*}(\overline{a}_{2n}\varpi_{2n}\phi_{2n} + \rho^-_{2n}\varpi_{2n-1}\phi^*_{2n-1}) = \\
&= B^e_{n*}\overline{a}_{2n}(\rho^+_{2n+1}\varpi_{2n+1}\phi_{2n+1} - a_{2n+1}\varpi_{2n}\phi^*_{2n}) + \\
&\quad + B^e_{n-1*}\rho^-_{2n}(\overline{a}_{2n-1}\varpi_{2n-1}\phi_{2n-1} + \rho^-_{2n-1}\varpi_{2n-2}\phi^*_{2n-2}) = \\
&= \overline{a}_{2n}\rho^+_{2n+1}\varpi_{2n+1}\chi_{2n+1} - \overline{a}_{2n}a_{2n+1}\varpi_{2n}\chi_{2n} + \\
&\quad + \overline{a}_{2n-1}\rho^-_{2n}\varpi_{2n-1}\chi_{2n-1} + \rho^-_{2n-1}\rho^-_{2n}\varpi_{2n-2}\chi_{2n-2},
\end{aligned}
\tag{34}
$$

while

$$\varpi^*_0\chi_0 = \rho^+_1\varpi_1\chi_1 - a_1\varpi_0\chi_0.$$

This is the five-term linear recurrence for $(\chi_n)_{n\geq 0}$, which can be written in the form

$$\begin{pmatrix} \chi_0(z) & \chi_1(z) & \cdots \end{pmatrix} \left(\varpi^*_{\mathcal{A}}(z) - \varpi_{\mathcal{A}}(z)\,\hat{\mathcal{C}}\right) = 0, \tag{35}$$

where $\hat{\mathcal{C}}$ is the five-diagonal matrix

$$\hat{\mathcal{C}} = \begin{pmatrix}
-a_1 & -\rho^-_1 a_2 & \rho^-_1\rho^-_2 & 0 & 0 & 0 & \cdots \\
\rho^+_1 & -\overline{a}_1 a_2 & \overline{a}_1\rho^-_2 & 0 & 0 & 0 & \cdots \\
0 & -\rho^+_2 a_3 & -\overline{a}_2 a_3 & -\rho^-_3 a_4 & \rho^-_3\rho^-_4 & 0 & \cdots \\
0 & \rho^+_2\rho^+_3 & \overline{a}_2\rho^+_3 & -\overline{a}_3 a_4 & \overline{a}_3\rho^-_4 & 0 & \cdots \\
0 & 0 & 0 & -\rho^+_4 a_5 & -\overline{a}_4 a_5 & -\rho^-_5 a_6 & \cdots \\
0 & 0 & 0 & \rho^+_4\rho^+_5 & \overline{a}_4\rho^+_5 & -\overline{a}_5 a_6 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}. \tag{36}$$

Using (11) we find that $\hat{\mathcal{C}}$ can be related to the unitary five-diagonal matrix $\mathcal{C}$ given in (6) by

$$\hat{\mathcal{C}} = \eta_{\mathcal{A}}^{-1} \mathcal{C}\, \eta_{\mathcal{A}} = \mathcal{C}_{\mathcal{A}}. \tag{37}$$

Bearing in mind (15) and (16), this relation implies that (35) is equivalent to

$$\begin{pmatrix} \chi_0(z) & \chi_1(z) & \cdots \end{pmatrix} \left(z - \tilde{\zeta}_{\mathcal{A}}(\mathcal{C})\right) = 0, \tag{38}$$

which shows that $\mathcal{M}$ is invariant under T_μ and $\tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ is the matrix representation of $T_\mu \restriction \mathcal{M}$ with respect to $(\chi_n)_{n\geq 0}$.

Similar arguments to those given in the proof of Theorem 4.1 prove that $T_\mu \restriction \mathcal{M}$ is unitary iff $\mathcal{M} = L^2_\mu$. However, $T_\mu \restriction \mathcal{M}$ is unitary whenever $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$ because in this case $\tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ is unitary for any sequence $\boldsymbol{a}$ in $\mathbb{D}$. Therefore, $\mathcal{M} = L^2_\mu$, i.e., the ORF $(\chi_n)_{n\geq 0}$ are a basis of L^2_μ, which implies that $\tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ is a matrix of the full operator T_μ.

□

Remark 5.2. We know that $(\chi_n)_{n\geq 0}$ and $(\chi_{n*})_{n\geq 0}$ are basis of L^2_μ at the same time, and the corresponding matrices of T_μ are related by transposition. Therefore, the previous theorem can be equivalently formulated saying that $(\chi_{n*})_{n\geq 0}$ is a basis of L^2_μ whenever $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$ and, in this case, the related matrix of T_μ is $\mathcal{U}^T$. Notice that the second equalitiy in (14) implies that $\mathcal{U}^T = \tilde{\zeta}_{\mathcal{A}}(\mathcal{C}^T)$ because $\mathcal{A}$ is diagonal.

Theorem 5.1 states that, contrary to the case of the ORF $(\phi_n)_{n\geq 0}$, $(\chi_n)_{n\geq 0}$ and $(\chi_{n*})_{n\geq 0}$ are basis of L^2_μ for any measure μ on $\mathbb{T}$ if $(\alpha_n)_{n\geq 1}$ is compactly included in $\mathbb{D}$. Indeed, the completeness of $(\chi_n)_{n\geq 0}$ and $(\chi_{n*})_{n\geq 0}$ in L^2_μ holds even under a more general condition for $\boldsymbol{\alpha}$, as the next theorem shows. Denoting $\mathcal{F}_* = \{f_* : f \in \mathcal{F}\}$ for any set $\mathcal{F}$ of complex functions, the problem is to find sufficient conditions for the equality $\mathcal{M} = L^2_\mu$ or, equivalently, $\mathcal{M}_* = L^2_\mu$.

Proposition 5.3. *Let $\boldsymbol{\alpha}$ be a sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $(\chi_n)_{n\geq 0}$ the ORF associated with $(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$. If*

$$\sum_{k=1}^{\infty}(1 - |\alpha_{2k-1}|) = \sum_{k=1}^{\infty}(1 - |\alpha_{2k}|) = \infty,$$

then $(\chi_n)_{n\geq 0}$ and $(\chi_{n*})_{n\geq 0}$ are both basis of L^2_μ.

Proof. Given an arbitrary sequence $\boldsymbol{\beta} = (\beta_n)_{n\geq 1}$ in $\overline{\mathbb{C}}\setminus\mathbb{T}$, let us use the notation $\mathcal{L}_\infty(\boldsymbol{\beta})$ for the set of rational functions with poles in $\hat{\boldsymbol{\beta}} = (\hat{\beta}_n)_{n\geq 1}$, counted with multiplicity, i.e.,

$$\mathcal{L}_\infty(\boldsymbol{\beta}) = \bigcup_{n\geq 2} \frac{\mathcal{P}_n}{\varpi_{\beta_1}\cdots\varpi_{\beta_{n-1}}}, \tag{39}$$

where $\varpi_\infty(z) = z$. Also, let $\mathcal{L}(\boldsymbol{\beta})$ be the closure of $\mathcal{L}_\infty(\boldsymbol{\beta})$ in L^2_μ. Notice that $\mathcal{L}(\boldsymbol{\beta})_* = \mathcal{L}(\hat{\boldsymbol{\beta}})$ and $\mathcal{M} = \mathcal{L}(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$.

We will show that

(*i*) $\sum_{k=1}^\infty (1-|\alpha_{2k-1}|) = \infty \;\Rightarrow\; \{z^j\}_{j\in\mathbb{N}} \subset \mathcal{L}(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$,

(*ii*) $\sum_{k=1}^\infty (1-|\alpha_{2k}|) = \infty \;\Rightarrow\; \{z^{-j}\}_{j\in\mathbb{N}} \subset \mathcal{L}(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$.

This demonstrates the proposition since $\operatorname{span}\{z^j\}_{j\in\mathbb{Z}}$ is dense in L^2_μ.

Indeed, we only must prove (*i*) since it implies (*ii*). To see this, assume that (*i*) holds for any sequence $(\alpha_n)_{n\geq 1}$ in $\mathbb{D}$. Then, applying (*i*) to the sequence $(\alpha_2, \alpha_1, \alpha_4, \alpha_3, \ldots)$ we find that $\sum_{k=1}^\infty (1-|\alpha_{2k}|) = \infty$ ensures $\{z^j\}_{j\in\mathbb{N}} \subset \mathcal{L}(\alpha_2, \hat{\alpha}_1, \alpha_4, \hat{\alpha}_3, \ldots) = \mathcal{L}(\hat{\alpha}_1, \alpha_2, \hat{\alpha}_3, \alpha_4, \ldots)$, which, applying the $*$-involution, becomes $\{z^{-j}\}_{j\in\mathbb{N}} \subset \mathcal{L}(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$.

The conditions $\sum_{k=1}^\infty (1-|\alpha_{2k-1}|) = \infty$ and $\sum_{n=1}^\infty (1-|\alpha_{2k}|) = \infty$ are equivalent respectively to the divergence (to zero) in $\mathbb{D}$ of the Blaschke products $B^o = \prod_{k=1}^\infty \zeta_{2k-1}$ and $B^e = \prod_{k=1}^\infty \zeta_{2k}$. Thus, all what we must prove is that the divergence of B^o implies that $z^j \in \mathcal{L}(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$ for any $j \in \mathbb{N}$.

According to (39), $\{z^j\}_{j\in\mathbb{N}} \subset \mathcal{L}_\infty(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots) = \mathcal{M}_\infty$ if $\alpha_{2k-1} = 0$ for infinitely many values $k \in \mathbb{N}$. Hence, we only need to study the opposite case that, without loss of generality, we can suppose is $\alpha_1 = \alpha_3 = \cdots = \alpha_{2s-1} = 0$ and $\alpha_{2k-1} \neq 0$ for $k > s$. Then, $\{z, \ldots, z^s\} \subset \mathcal{M}_\infty$ and, since $\langle f, f\rangle_\mu \leq \|f\|^2_\infty$ for any $f \in \mathcal{M}_\infty$, it suffices to prove that

$$\inf_{f\in\mathcal{M}_n} \|z^j - f(z)\|_\infty \xrightarrow{n} 0, \qquad \forall j > s.$$

To measure the L^∞-distance between a polynomial and a subspace like $\mathcal{M}_n$ we can use the following result (see [1, p. 243] or the more recent reference [10, p. 150]):

$$\min_{q\in\mathcal{P}_N} \left\| \frac{z^N + q(z)}{(z-w_1)\cdots(z-w_n)} \right\|_\infty = \prod_{k=1}^n \frac{1}{\max\{|w_k|, 1\}}, \quad w_k \in \mathbb{C}, \quad N \geq n.$$

Therefore, if B^o diverges, taking $n > s$,

$$\inf_{\substack{f \in \mathcal{M}_{2n} \\ a_k \in \mathbb{C}}} \left\| z^{s+m} + a_{m-1} z^{s+m-1} + \cdots + a_1 z^{s+1} - f(z) \right\|_\infty =$$

$$= \inf_{q \in \mathcal{P}_{2n+m-1}} \left\| \frac{z^{2n+m-1} + q(z)}{\prod_{k=1}^{n-1}(z-\alpha_{2k}) \prod_{k=s+1}^{n}(z-\hat{\alpha}_{2k-1})} \right\|_\infty = \prod_{k=s+1}^{n} |\alpha_{2k-1}| \xrightarrow{n} 0$$

and

$$\inf_{\substack{f \in \mathcal{M}_{2n+1} \\ a_k \in \mathbb{C}}} \left\| z^{s+m} + a_{m-1} z^{s+m-1} + \cdots + a_1 z^{s+1} - f(z) \right\|_\infty =$$

$$= \inf_{q \in \mathcal{P}_{2n+m}} \left\| \frac{z^{2n+m} + q(z)}{\prod_{k=1}^{n}(z-\alpha_{2k}) \prod_{k=s+1}^{n}(z-\hat{\alpha}_{2k-1})} \right\|_\infty = \prod_{k=s+1}^{n} |\alpha_{2k-1}| \xrightarrow{n} 0$$

for any $m \in \mathbb{N}$. This result implies by induction on m that $z^{s+m} \in \mathcal{M}$ for any $m \in \mathbb{N}$.

□

In the polynomial case $\mathcal{A} = 0$ so $\mathcal{U} = \mathcal{C}$ becomes a five-diagonal matrix. However, in the general case $\mathcal{U}$ is not a band matrix but its Möbius transform $\zeta_{\mathcal{A}}(\mathcal{U}) = \mathcal{C}$ is five-diagonal. This fact makes the rational case more complicated than the polynomial one but, as we will see later, the matrix $\mathcal{U}$ can be also used in the rational case to transcribe certain properties of the measure μ into properties of the corresponding sequence $\boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$.

As in the Hessenberg case, the previous theorem provides a spectral interpretation of the support of the measure μ. The arguments are similar to those given in the proof of Theorem 4.2, but now the restriction $\log \mu' \notin L^1_m$ is not necessary. Hence, we obtain the following result.

Theorem 5.4. *Let $\boldsymbol{\alpha}$ be a sequence compactly included in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geq 0}$ the corresponding ORF and $(\chi_n)_{n\geq 0}$ the ORF associated with $(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$. If*

$$\mathcal{U} = \tilde{\zeta}_{\mathcal{A}}(\mathcal{C}), \qquad \mathcal{A} = \mathcal{A}(\boldsymbol{\alpha}), \qquad \mathcal{C} = \mathcal{C}(\boldsymbol{a}), \qquad \boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu),$$

and $\mathcal{E}$ is the spectral measure of $\mathcal{U}$, then $\mu = \mathcal{E}_{1,1}$. Besides, $\operatorname{supp}\mu = \sigma(\mathcal{U})$ and the mass points of μ are the eigenvalues of $\mathcal{U}$, which have geometric multiplicity 1. λ is a mass point iff $(\chi_n(\lambda))_{n\geq 0} \in \ell^2$. Given a mass point λ, the corresponding eigenvectors of $\mathcal{U}$ are spanned by $\begin{pmatrix} \chi_0(\lambda) & \chi_1(\lambda) & \cdots \end{pmatrix}^\dagger$ and $\mu(\{\lambda\}) = \left(\sum_{n=0}^\infty |\chi_n(\lambda)|^2\right)^{-1} = \left(\sum_{n=0}^\infty |\phi_n(\lambda)|^2\right)^{-1}$.

Analogously to the Hessenberg case, we can formulate the above spectral results in terms of pairs of band operators. Theorem 4.2 implies that $\operatorname{supp}\mu = \sigma(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{C}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{C}))$ and the mass points of μ are the eigenvalues of the five-diagonal pair $(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{C}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{C}))$. Also, given a mass point λ, the left eigenvectors of $(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{C}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{C}))$ are spanned by $(\,\chi_0(\lambda) \quad \chi_1(\lambda) \quad \cdots\,)\,\eta_{\mathcal{A}}^{-1/2}$. Furthermore, the factorization $\mathcal{C} = \mathcal{C}_o\mathcal{C}_e$ makes possible to formulate the above results using the tridiagonal pair

$$(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{C}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{C}))\,\mathcal{C}_e^\dagger = (\mathcal{C}_o + \mathcal{A}\mathcal{C}_e^\dagger, \mathcal{C}_e^\dagger + \mathcal{A}^\dagger\mathcal{C}_o)$$

instead of the five-diagonal pair.

5.1 Zeros of ORF and five-diagonal matrices

The previous results suggest that it should be possible a spectral interpretation of the zeros of ORF and PORF in terms of five-diagonal matrices. Similarly to the Hessenberg case, an important ingredient for this is the orthogonal truncation of $\zeta_n(T_\mu)$ on $\mathcal{M}_n$. Taking into account that $\mathcal{M}_n = B^e_{l*}\mathcal{L}_n$ with $l = [(n-1)/2]$, the following generalization of Theorem 4.3 is of interest to relate this truncation to the zeros of the ORF ϕ_n. This generalization deals with the orthogonal truncation of $\zeta_n(T_\mu)$ on $h\mathcal{L}_n$, $h \in L^2_\mu$, which is given by $\zeta_n(T_\mu)^{(h\mathcal{L}_n)} = L^h_n\zeta_n(T_\mu) \restriction h\mathcal{L}_n$, where $L^h_n\colon L^2_\mu \to L^2_\mu$ is the orthogonal projection on $h\mathcal{L}_n$.

Theorem 5.5. *Let $\boldsymbol{\alpha}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $\phi_n = p_n/\pi_n$ the related n-th ORF. Then, for any Borel function $h\colon \mathbb{T} \to \mathbb{T}$:*

1. *If Z_n is the set of zeros of ϕ_n, $\zeta_n(Z_n)$ is the set of eigenvalues of $\zeta_n(T_\mu)^{(h\mathcal{L}_n)}$, and these eigenvalues have geometric multiplicity 1.*
2. *If $p_n(z) \propto \prod_{k=1}^n (z-\lambda_k)$, the characteristic polynomial of $\zeta_n(T_\mu)^{(h\mathcal{L}_n)}$ is*

$$\prod_{k=1}^n (z - \zeta_n(\lambda_k)).$$

Proof. The operator multiplication by h in L^2_μ

$$h(T_\mu)\colon \underset{f \longrightarrow hf}{L^2_\mu \to L^2_\mu}$$

is unitary because h maps $\mathbb{T}$ on itself. When restricted in the following way

$$V \colon \mathcal{L}_n \to h\mathcal{L}_n \\ \quad f \longrightarrow hf$$

it yields a isometric isomorphism V between $\mathcal{L}_n$ and $h\mathcal{L}_n$. The orthogonal projection on $h\mathcal{L}_n$ is $L_n^h = h(T_\mu)L_n h(T_\mu)^\dagger$, where L_n is the orthogonal projection on $\mathcal{L}_n$. Thus, the orthogonal truncations of $\zeta_n(T_\mu)$ on $h\mathcal{L}_n$ and $\mathcal{L}_n$ are related by $\zeta_n(T_\mu)^{(h\mathcal{L}_n)} = V\zeta_n(T_\mu)^{(\mathcal{L}_n)}V^{-1}$, so they are unitarily equivalent. In consequence, they have the same eigenvalues and with the same geometric and algebraic multiplicity. Hence, the result follows from Theorem 4.3.

□

Taking $h = B_{l*}^e$, $l = [(n-1)/2]$, in the previous theorem we find that it holds for the orthogonal truncation of $\zeta_n(T_\mu)$ on $\mathcal{M}_n$, i.e., $\zeta_n(T_\mu)^{(\mathcal{M}_n)} = M_n\zeta_n(T_\mu) \restriction \mathcal{M}_n$, where $M_n \colon L_\mu^2 \to L_\mu^2$ is the orthogonal projection on $\mathcal{M}_n$. To give a matrix version of this result we simply need a matrix representation of $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$.

Theorem 5.6. *Let $\boldsymbol{\alpha}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $(\chi_n)_{n\geq 0}$ the ORF associated with $(\alpha_1, \hat\alpha_2, \alpha_3, \hat\alpha_4, \ldots)$. If $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$ and $\mathcal{C} = \mathcal{C}(\boldsymbol{a})$ with $\boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$, the matrix of $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$ with respect to $(\chi_k)_{k=0}^{n-1}$ is $\zeta_n(\mathcal{U}^{(n)})$, where*

$$\mathcal{U}^{(n)} = \tilde\zeta_{\mathcal{A}_n}(\mathcal{C}_n), \qquad \|\mathcal{U}^{(n)}\| = 1.$$

Proof. From the factorization $\mathcal{C} = \mathcal{C}_o\mathcal{C}_e$ we find that $\mathcal{C}_n = \mathcal{C}_{on}\mathcal{C}_{en}$. Only one among the factors $\mathcal{C}_{on}$ and $\mathcal{C}_{en}$ is unitary, but the norm of the remaining factor is 1, so $\|\mathcal{C}_n\| = 1$. Since $\tilde\zeta_{\mathcal{A}_n}$ leaves $\mathbb{T}_{\mathbb{C}^n}$ invariant, $\mathcal{U}^{(n)} = \tilde\zeta_{\mathcal{A}_n}(\mathcal{C}_n)$ is a well defined matrix with $\|\mathcal{U}^{(n)}\| = 1$. The same holds for $\zeta_n(\mathcal{U}^{(n)})$.

To prove that $\zeta_n(\mathcal{U}^{(n)})$ is the matrix representation of $\zeta_n(T_\mu)$ with respect to $(\chi_k)_{k=0}^{n-1}$, let us consider first an odd n. Then, the first n equations of (35) can be written as

$$\begin{pmatrix} \chi_0(z) & \cdots & \chi_{n-1}(z) \end{pmatrix} \left(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\,\hat{\mathcal{C}}_n\right) = b_n\varpi_n(z)\chi_n(z), \quad b_n \in \mathbb{C}^n.$$

(37) gives $\hat{\mathcal{C}}_n = \eta_{\mathcal{A}_n}^{-1}\mathcal{C}_n\eta_{\mathcal{A}_n} = (\mathcal{C}_n)_{\mathcal{A}_n}$. This, together with identities (15) and (16), yields

$$\begin{pmatrix} \chi_0(z) & \cdots & \chi_{n-1}(z) \end{pmatrix} \left(z - \mathcal{U}^{(n)}\right) = c_n\varpi_n(z)\chi_n(z), \quad c_n \in \mathbb{C}^n.$$

Using (19) we get

$$\begin{pmatrix} \chi_0(z) & \cdots & \chi_{n-1}(z) \end{pmatrix} \big(\zeta_n(z) - \zeta_n(\mathcal{U}^{(n)})\big) = d_n \chi_n(z), \quad d_n \in \mathbb{C}^n,$$

and, taking into account that

$$M_n \chi_k = \begin{cases} \chi_k & \text{if } k < n, \\ 0 & \text{if } k \geq n, \end{cases}$$

we finally obtain

$$\begin{pmatrix} M_n \zeta_n \chi_0 & \cdots & M_n \zeta_n \chi_{n-1} \end{pmatrix} = \begin{pmatrix} \chi_0 & \cdots & \chi_{n-1} \end{pmatrix} \zeta_n(\mathcal{U}^{(n)}).$$

This equality proves that $\zeta_n(\mathcal{U}^{(n)})$ is the matrix of $M_n \zeta_n(T_\mu) \restriction \mathcal{M}_n$ with respect to $(\chi_k)_{k=0}^{n-1}$.

On the other hand, if n is even, we consider the orthogonal truncation of $\zeta_n(T_\mu)$ on $\mathcal{M}_{n*}$, i.e., $\zeta_n(T_\mu)^{(\mathcal{M}_{n*})} = M_{n*} \zeta_n(T_\mu) \restriction \mathcal{M}_{n*}$, where $M_{n*} \colon L^2_\mu \to L^2_\mu$ is the orthogonal projection on $\mathcal{M}_{n*}$. Taking into account that $\mathcal{C}$ is unitary, the identity obtained by applying the $*$-involution on (35) reads

$$\begin{pmatrix} \chi_{0*}(z) & \chi_{1*}(z) & \cdots \end{pmatrix} \big(\varpi^*_{\mathcal{A}}(z) - \varpi_{\mathcal{A}}(z)\, (\mathcal{C}^T)_{\mathcal{A}}\big) = 0. \tag{40}$$

A similar reasoning starting from the first n equations of this equality proves that the matrix of $\zeta_n(T_\mu)^{(\mathcal{M}_{n*})}$ with respect to $(\chi_{k*})_{k=0}^{n-1}$ is $\zeta_n(\mathcal{U}^{(n)}_*)$, where $\mathcal{U}^{(n)}_* = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{C}^T_n)$. Notice that (14) implies that $\mathcal{U}^{(n)}_* = \mathcal{U}^{(n)T}$ because $\mathcal{A}_n$ is diagonal.

Given a measure μ on $\mathbb{T}$, for any $n \in \mathbb{N}$, the subspace $\mathcal{M}_n$ only depends on the parameters $\alpha_1, \dots, \alpha_{n-1}$ of the sequence $\boldsymbol{\alpha}$, so the same holds for the orthogonal truncations $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$ and $\zeta_n(T_\mu)^{(\mathcal{M}_{n*})}$. Therefore, concerning the spectral properties of these truncations we can suppose without loss of generality that $\boldsymbol{\alpha}$ is compactly supported on $\mathbb{D}$. Then, the matrix representations of T_μ with respect to $(\chi_k)_{k\geq 0}$ and $(\chi_{k*})_{k \geq 0}$ are $\mathcal{U}$ and $\mathcal{U}^T$ respectively. Hence, the representations of the orthogonal truncations $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$ and $\zeta_n(T_\mu)^{(\mathcal{M}_{n*})}$ with respect to $(\chi_k)_{k=0}^{n-1}$ and $(\chi_{k*})_{k=0}^{n-1}$ are the principal submatrices $(\zeta_n(\mathcal{U}))_n$ and $(\zeta_n(\mathcal{U}^T))_n$ respectively. The fact that $(\zeta_n(\mathcal{U}^T))_n = (\zeta_n(\mathcal{U}))_n^T$ implies that, when n is even, the matrix of $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$ with respect to $(\chi_k)_{k=0}^{n-1}$ is $\zeta_n(\mathcal{U}^{(n)}_*)^T = \zeta_n(\mathcal{U}^{(n)})$.

□

Remark 5.7. The proof of the previous theorem also shows that the matrix of the orthogonal truncation $\zeta_n(T_\mu)^{(\mathcal{M}_{n*})}$ with respect to $(\chi_{k*})_{k=0}^{n-1}$ is $\zeta_n(\mathcal{U}^{(n)})^T$.

As a consequence of Theorems 5.5 and 5.6 we have the following spectral interpretation of the zeros of ORF in terms of Möbius transformations of five-diagonal matrices.

Theorem 5.8. *Let $\boldsymbol{\alpha}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geq 0}$ the corresponding ORF and $(\chi_n)_{n\geq 0}$ the ORF associated with the sequence $(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$. Let $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$ and $\mathcal{C} = \mathcal{C}(\boldsymbol{a})$ with $\boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. If $\mathcal{U}^{(n)} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{C}_n)$, then:*

1. *The zeros of ϕ_n are the eigenvalues of $\mathcal{U}^{(n)}$, which have geometric multiplicity 1. If λ is a zero of ϕ_n, the related left and right eigenvectors of $\mathcal{U}^{(n)}$ are spanned by $X_n(\lambda)$ and $Y_n(\lambda)^T$ respectively, where*
$$X_n = B^e_{[\frac{n-1}{2}]} \begin{pmatrix} \chi_0 & \cdots & \chi_{n-1} \end{pmatrix}, \qquad Y_n = B^o_{[\frac{n}{2}]} \begin{pmatrix} \chi_{0*} & \cdots & \chi_{n-1*} \end{pmatrix}.$$
2. $\phi_n = \dfrac{p_n}{\pi_n}$ *with p_n proportional to the characteristic polynomial of $\mathcal{U}^{(n)}$.*

Proof. The vectors $X_n(z)$ and $Y_n(z)$ are rational functions with the poles lying on $\mathbb{E}$, so they can be evaluated at any zero λ of ϕ_n since $\lambda \in \mathbb{D}$. Besides, $X_n(\lambda), Y_n(\lambda) \neq 0$ because $B^e_k \chi_{2k} = \phi^*_{2k}$, $B^o_k \chi_{2k-1*} = \phi^*_{2k-1}$ and ϕ^*_n has its zeros in $\mathbb{E}$.

The proof of the theorem is similar to the case of Theorem 4.5, the only difference concerning the identification of the eigenvectors. To obtain the left eigenvectors of $\mathcal{U}^{(n)}$ let us consider the first n equations of (35) for an arbitrary $n \in \mathbb{N}$. These equations can be written as

$$\begin{aligned} \begin{pmatrix} \chi_0(z) & \cdots & \chi_{n-1}(z) \end{pmatrix} \Big(\varpi^*_{\mathcal{A}_n}(z) - \varpi_{\mathcal{A}_n}(z)\, \hat{\mathcal{C}}_n\Big) = \\ = b_n \varpi_n(z)\chi_n(z) + d_n \varpi_{n+1}(z)\chi_{n+1}(z), \end{aligned} \tag{41}$$

where $b_n, d_n \in \mathbb{C}^n$ are

$$b_n = \begin{cases} \rho^+_n \begin{pmatrix} 0 & \cdots & 0 & \rho^+_{n-1} & \overline{a}_{n-1} \end{pmatrix} & \text{odd } n, \\ -\rho^+_n a_{n+1} \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} & \text{even } n, \end{cases}$$
$$d_n = \begin{cases} 0 & \text{odd } n, \\ \rho^+_n \rho^+_{n+1} \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} & \text{even } n. \end{cases}$$

Writing χ_n and χ_{n*} in terms of ϕ_n and ϕ_n^* with the aid of (32) and (33), and using the first equation of (12) in the case of even n, (41) reads

$$\begin{gathered} (\,\chi_0(z) \;\; \cdots \;\; \chi_{n-1}(z)\,)\left(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\,\hat{\mathcal{C}}_n\right) = \rho_n^+\varpi_n(z)B_{l*}^e(z)\phi_n(z)v_n, \\ l = \left[\tfrac{n-1}{2}\right], \quad v_n \in \mathbb{C}^n, \quad v_n = \begin{cases} (\,0 \;\; \cdots \;\; 0 \;\; \rho_{n-1}^+ \;\; \overline{a}_{n-1}\,) & \text{odd } n, \\ (\,0 \;\; \cdots \;\; 0 \;\; 1\,) & \text{even } n. \end{cases} \end{gathered} \tag{42}$$

Also, remember that (15), (16) and $\hat{\mathcal{C}}_n = \eta_{\mathcal{A}_n}^{-1}\mathcal{C}_n\eta_{\mathcal{A}_n}$ imply that

$$(z - \mathcal{U}^{(n)})\,\tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{C}}_n) = \varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\,\hat{\mathcal{C}}_n. \tag{43}$$

Therefore, if λ is a zero of ϕ_n, (42) and (43) show that $X_n(\lambda)$ is a left eigenvector of $\mathcal{U}^{(n)}$ with eigenvalue λ.

Proceeding in a similar way with the first n equations of (40) we find that $Y_n(\lambda)$ is a left eigenvector of $\mathcal{U}^{(n)T}$ with eigenvalue λ for any zero λ of ϕ_n. Therefore, $Y_n(\lambda)^T$ is a right eigenvector of $\mathcal{U}^{(n)}$.

□

For a unitary matrix, like $\mathcal{V}$ in the case $\boldsymbol{a} \notin \ell^2$, $\mathcal{V}^{(n;u)}$ or $\mathcal{U}$, left and right eigenvectors are related by the †-operation. However, this is not the case of the matrices $\mathcal{V}^{(n)}$ or $\mathcal{U}^{(n)}$. Theorem 4.5 only gives information about the left eigenvectors of $\mathcal{V}^{(n)}$, while Theorem 5.8 provides both, the left and right eigenvectors of $\mathcal{U}^{(n)}$. Apart from the simplest form of $\mathcal{U}^{(n)}$, this is another advantage of using this matrix instead of $\mathcal{V}^{(n)}$ for the spectral representation of the zeros of ORF.

Concerning the form of the eigenvectors of $\mathcal{U}^{(n)}$, notice that the factors $B_{[(n-1)/2]}^e$ and $B_{[n/2]}^o$ in X_n and Y_n are necessary to avoid any problem when evaluating them on a point of $\mathbb{D}$. However, if a zero λ of ϕ_n does not coincide with any α_k for $k = 1, \dots, n-1$, then we can take as left and right eigenvectors $(\,\chi_0(\lambda), \dots, \chi_{n-1}(\lambda)\,)$ and $(\,\chi_{0*}(\lambda), \dots, \chi_{n-1*}(\lambda)\,)^T$ respectively.

As in the Hessenberg case, there are other alternatives to express p_n as a determinant. Indeed, from (15), (16) and the identity $\hat{\mathcal{C}}_n = \eta_{\mathcal{A}_n}^{-1}\mathcal{C}_n\eta_{\mathcal{A}_n}$,

$$p_n(z) \propto \det\left(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\,\mathcal{C}_n\right) = \det\left(z\,\tilde{\varpi}_{\mathcal{A}_n}(\mathcal{C}_n) - \tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{C}_n)\right).$$

So p_n can be calculated as a determinant of a five-diagonal matrix. Furthermore, the last expression shows that the zeros of ϕ_n are the eigenvalues of the five-diagonal pair $(\tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{C}_n), \tilde{\varpi}_{\mathcal{A}_n}(\mathcal{C}_n))$. The associated left eigenvectors with eigenvalue λ are spanned by $X_n(\lambda)\,\eta_{\mathcal{A}_n}^{-1/2}$.

Besides, the factorization $\mathcal{C}_n = \mathcal{C}_{on}\mathcal{C}_{en}$ permits us to express p_n as a determinant of a tridiagonal matrix. If n is odd, $\mathcal{C}_{en}$ is unitary, thus

$$p_n(z) \propto \det\left(z(\mathcal{C}_{en}^\dagger + \mathcal{A}_n^\dagger \mathcal{C}_{on}) - (\mathcal{C}_{on} + \mathcal{A}_n \mathcal{C}_{en}^\dagger)\right)$$

and the zeros of ϕ_n are the eigenvalues of the tridiagonal pair

$$(\mathcal{C}_{on} + \mathcal{A}_n \mathcal{C}_{en}^\dagger, \mathcal{C}_{en}^\dagger + \mathcal{A}_n^\dagger \mathcal{C}_{on}),$$

which has the same left eigenvectors as $(\tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{C}_n), \tilde{\varpi}_{\mathcal{A}_n}(\mathcal{C}_n))$. On the contrary, $\mathcal{C}_{on}$ is unitary for an even n. In this situation we can use the fact that $\mathcal{U}^{(n)}$ and $\mathcal{U}^{(n)T} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{C}_n^T)$ have the same characteristic polynomial, and the left eigenvectors of one of them are the transposed of the right eigenvectors of the other one. Hence,

$$p_n(z) \propto \det\left(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\, \mathcal{C}_n^T\right) = \det\left(z\, \tilde{\varpi}_{\mathcal{A}_n}(\mathcal{C}_n^T) - \tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{C}_n^T)\right)$$

and, bearing in mind that $\mathcal{C}_n^T = \mathcal{C}_{en}\mathcal{C}_{on}$,

$$p_n(z) \propto \det\left(z(\mathcal{C}_{on}^\dagger + \mathcal{A}_n^\dagger \mathcal{C}_{en}) - (\mathcal{C}_{en} + \mathcal{A}_n \mathcal{C}_{on}^\dagger)\right).$$

So, the zeros of ϕ_n are the eigenvalues of the tridiagonal pair

$$(\mathcal{C}_{en} + \mathcal{A}_n \mathcal{C}_{on}^\dagger, \mathcal{C}_{on}^\dagger + \mathcal{A}_n^\dagger \mathcal{C}_{en})$$

and the left eigenvectors with eigenvalue λ are spanned by $Y_n(\lambda)\, \eta_{\mathcal{A}_n}^{-1/2}$.

The zeros of the PORF Q_n^v have a spectral interpretation in terms of band matrices too. Such an interpretation has to do with the matrix representation of $T_{\mu_n^v}$ with respect $(\chi_k)_{k=0}^{n-1}$, which is an orthonormal basis of $L^2_{\mu_n^v}$ due to the exactness of the quadrature formulas associated with μ_n^v. Similar arguments to those appearing before Theorem 4.6 show that the zeros of the PORF should be related to the unitary matrix $\mathcal{C}_n^u$ obtained from $\mathcal{C}_n$ when substituting the parameter $a_n \in \mathbb{D}$ by $u \in \mathbb{T}$. More precisely, we have the following result, which can be understood as a limit case of Theorems 5.6 and 5.8.

Theorem 5.9. *Let $\boldsymbol{\alpha}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geq 0}$ the corresponding ORF, $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$ and $\mathcal{C} = \mathcal{C}(\boldsymbol{a})$ with $\boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. If $Q_n^v = \phi_n + v\phi_n^*$ is the n-th PORF related to $v \in \mathbb{T}$ and μ_n^v is the associated measure, then:*

1. *The matrix of $T_{\mu_n^v}$ with respect to $(\chi_k)_{k=0}^{n-1}$ is*

$$\mathcal{U}^{(n;u)} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{C}_n^u), \qquad u = \tilde{\zeta}_{a_n}(v).$$

2. *The zeros of Q_n^v are the eigenvalues of $\mathcal{U}^{(n;u)}$. If λ is a zero of Q_n^v, the related eigenvectors of $\mathcal{U}^{(n;u)}$ are spanned by $(\, \chi_0(\lambda) \quad \cdots \quad \chi_{n-1}(\lambda)\,)^\dagger$.*

3. $Q_n^v = \dfrac{q_n^v}{\pi_n}$ *with q_n^v proportional to the characteristic polynomial of $\mathcal{U}^{(n;u)}$.*

Proof. As in the case of Theorem 4.6, it suffices to prove item 1. For an odd $n = 2l+1$, using (30) in a similar computation to that of (34) gives

$$\begin{aligned}
\varpi_{n-2}^*\chi_{n-2} &= \rho_{n-1}^+\rho_n^+\varpi_n B_{[n/2]*}^e \frac{Q_n^v}{1+\overline{a}_n v} - \rho_{n-1}^+ u\, \varpi_{n-1}\chi_{n-1} - \\
&\quad - \overline{a}_{n-2}a_{n-1}\varpi_{n-2}\chi_{n-2} - \rho_{n-2}^- a_{n-1}\varpi_{n-3}\chi_{n-3}, \\
\varpi_{n-1}^*\chi_{n-1} &= \overline{a}_{n-1}\rho_n^+\varpi_n B_{[n/2]*}^e \frac{Q_n^v}{1+\overline{a}_n v} - \overline{a}_{n-1} u\, \varpi_{n-1}\chi_{n-1} + \\
&\quad + \overline{a}_{n-2}\rho_{n-1}^-\varpi_{n-2}\chi_{n-2} + \rho_{n-2}^-\rho_{n-1}^-\varpi_{n-3}\chi_{n-3},
\end{aligned}$$

where $u = \tilde{\zeta}_{a_n}(v)$. These relations can be combined with the first $n-2$ equations of (35) in the matrix identity

$$\begin{aligned}
(\, \chi_0(z) \ \cdots \ \chi_{n-1}(z)\,) \left(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\, \hat{\mathcal{C}}_n^u\right) &= \\
&= b_n\varpi_n(z)B_{l*}^e(z)\, Q_n^v(z), \quad b_n \in \mathbb{C}^n,
\end{aligned}$$

with $\hat{\mathcal{C}}_n^u = \eta_{\mathcal{A}_n}^{-1}\mathcal{C}_n^u\eta_{\mathcal{A}_n} = (\mathcal{C}_n^u)_{\mathcal{A}_n}$. Thus, using (15) and (16) we find that

$$(\, \chi_0(z) \ \cdots \ \chi_{n-1}(z)\,)\left(z - \mathcal{U}^{(n;u)}\right) = c_n\varpi_n(z)B_{l*}^e(z)\, Q_n^v(z), \quad c_n \in \mathbb{C}^n.$$

Therefore, $\mathcal{U}^{(n;u)}$ is the matrix of $T_{\mu_n^v}$ with respect to $(\chi_k)_{k=0}^{n-1}$ because $Q_n^v = 0$ in $L^2_{\mu_n^v}$.

On the other hand, if $n = 2l$ is even, proceeding in a similar way with (30) and (35) we arrive at

$$(\, \chi_{0*}(z) \ \cdots \ \chi_{n-1*}(z)\,)\left(z - \mathcal{U}^{(n;u)^T}\right) = c_n\varpi_n(z)B_{l*}^o(z)\, Q_n^v(z), \quad c_n \in \mathbb{C}^n,$$

so $\mathcal{U}^{(n;u)^T}$ is the matrix of $T_{\mu_n^v}$ with respect to $(\chi_{k*})_{k=0}^{n-1}$. Consequently, the matrix of $T_{\mu_n^v}$ with respect to $(\chi_k)_{k=0}^{n-1}$ is $\mathcal{U}^{(n;u)}$.

□

The zeros of a PORF can be also interpreted as eigenvalues of a pair of band matrices. If $u = \tilde\zeta_{a_n}(v)$,

$$q_n^v(z) \propto \det\left(\varpi^*_{\mathcal{A}_n}(z) - \varpi_{\mathcal{A}_n}(z)\,\mathcal{C}_n^u\right) = \det\left(z\,\tilde\varpi_{\mathcal{A}_n}(\mathcal{C}_n^u) - \tilde\varpi^*_{\mathcal{A}_n}(\mathcal{C}_n^u)\right)$$

gives q_n^v as a determinant of a five-diagonal matrix. The zeros of Q_n^v are the eigenvalues of the five-diagonal pair $(\tilde\varpi^*_{\mathcal{A}_n}(\mathcal{C}_n^u), \tilde\varpi_{\mathcal{A}_n}(\mathcal{C}_n^u))$ and, given an eigenvalue λ, $(\,\chi_0(\lambda) \quad \cdots \quad \chi_{n-1}(\lambda)\,)\,\eta_{\mathcal{A}_n}^{-1/2}$ spans the corresponding left eigenvectors subspace.

We have also a factorization $\mathcal{C}_n^u = \mathcal{C}_{on}^u \mathcal{C}_{en}^u$, where $\mathcal{C}_{on}^u$ and $\mathcal{C}_{en}^u$ are the result of substituting a_n by u in $\mathcal{C}_{on}$ and $\mathcal{C}_{en}$ respectively (this substitution actually takes place only in $\mathcal{C}_{on}$ or $\mathcal{C}_{en}$, depending whether n is odd or even). $\mathcal{C}_{on}^u$ and $\mathcal{C}_{en}^u$ are both unitary, so

$$p_n(z) \propto \det\left(z(\mathcal{C}_{en}^{u\dagger} + \mathcal{A}_n^\dagger \mathcal{C}_{on}^u) - (\mathcal{C}_{on}^u + \mathcal{A}_n \mathcal{C}_{en}^{u\dagger})\right)$$

and the zeros of ϕ_n are the eigenvalues of the tridiagonal pair

$$(\mathcal{C}_{on}^u + \mathcal{A}_n \mathcal{C}_{en}^{u\dagger}, \mathcal{C}_{en}^{u\dagger} + \mathcal{A}_n^\dagger \mathcal{C}_{on}^u),$$

which has the same left eigenvectors as $(\tilde\varpi^*_{\mathcal{A}_n}(\mathcal{C}_n^u), \tilde\varpi_{\mathcal{A}_n}(\mathcal{C}_n^u))$.

6 Applications

In this section we will present some applications of the spectral theory previously developed for the ORF on the unit circle. We will use the results involving five-diagonal matrices due to their advantages. The corresponding spectral theory associates with each sequence of ORF a five-diagonal unitary matrix $\mathcal{C}(\boldsymbol{a})$ depending on the parameters $\boldsymbol{a} = (a_n)_{n\geq 1}$ of the recurrence relation, and a diagonal matrix $\mathcal{A}(\boldsymbol{\alpha})$ depending on the sequence $\boldsymbol{\alpha} = (\alpha_n)_{n\geq 1}$ which defines the poles $\hat\alpha_n$. These band matrices keep all the information about the ORF since they generate the full sequence of ORF through the associated recurrence. The importance of these matrices is that they play the role of a simple short cut that connects directly the parameters $\boldsymbol{a}$, $\boldsymbol{\alpha}$ to the ORF and the related orthogonality measure.

An essential difference with the polynomial case is that the matrix directly related to the ORF and the orthogonality measure is not the five-diagonal one, but an operator Möbius transform of it, namely, $\mathcal{U}(\boldsymbol{a}, \boldsymbol{\alpha}) = \tilde\zeta_{\mathcal{A}(\boldsymbol{\alpha})}(\mathcal{C}(\boldsymbol{a}))$.

This introduces important difficulties when trying to apply the spectral theory to the rational case. However, in spite of these difficulties, the matrix tool $\mathcal{U}(\boldsymbol{a},\boldsymbol{\alpha})$ becomes powerful enough to deal with hard problems even in the rational case. To understand the scope of the rational spectral theory, we will use it to solve some non trivial problems about the relation between the behavior of the sequences $\boldsymbol{a}$, $\boldsymbol{\alpha}$ and the properties of the corresponding orthogonality measure $\mu(\boldsymbol{a},\boldsymbol{\alpha})$. The answers to these problems are known for OP, but the generalizations to ORF are new.

The strategy will be to apply standard results of perturbation theory to the unitary operator on ℓ^2 defined by the matrix $\mathcal{U}(\boldsymbol{a},\boldsymbol{\alpha})$. We will apply such perturbation results to the comparison of $\mathcal{U}(\boldsymbol{a},\boldsymbol{\alpha})$ with another normal matrix, eventually with the form $\mathcal{U}(\boldsymbol{b},\boldsymbol{\beta})$. A useful remark for these comparisons is that, for $\boldsymbol{\beta}$ compactly supported in $\mathbb{D}$, $\mathcal{U}(\boldsymbol{b},\boldsymbol{\beta})$ defines a unitary operator for any sequence $\boldsymbol{b}$ in $\overline{\mathbb{D}}$ since, then, $\mathcal{C}(\boldsymbol{b})$ is unitary. However, $\mathcal{U}(\boldsymbol{b},\boldsymbol{\beta})$ only represents a multiplication operator on $\mathbb{T}$ when $\boldsymbol{b}$ lies on $\mathbb{D}$. When $b_n \in \mathbb{T}$ for some n we know that $\mathcal{C}(\boldsymbol{b})$ decomposes as a direct sum of an $n \times n$ and an infinite matrix. Taking into account that $\mathcal{A}(\boldsymbol{\beta})$ is diagonal, a similar decomposition holds for $\mathcal{U}(\boldsymbol{b},\boldsymbol{\beta})$.

The results of operator theory that we will apply state that two operators T, S on H have some common spectral property provided that the perturbation $T-S$ belongs to certain class of operators. We will deal with two kinds of perturbations: compact and trace class operators. Both are subsets of $\mathbb{B}_H$ that are closed under sum, left and right product by any element of $\mathbb{B}_H$ and also under the $\dagger$-operation, that is, they are hermitian ideals of $\mathbb{B}_H$. This fact is the key that permit us to use techniques of band matrices in the spectral theory of ORF, according to the following result.

Proposition 6.1. *Let $\mathfrak{I}$ be a hermitian ideal of $\mathbb{B}_H$. If $A, B \in \mathbb{D}_H$ are normal and $AB = BA$, the condition $A - B \in \mathfrak{I}$ implies the equivalences*

$$T - S \in \mathfrak{I} \;\Leftrightarrow\; \zeta_A(T) - \zeta_B(S) \in \mathfrak{I} \;\Leftrightarrow\; \tilde{\zeta}_A(T) - \tilde{\zeta}_B(S) \in \mathfrak{I}, \quad \forall\, T, S \in \overline{\mathbb{D}}_H.$$

Proof. It suffices to prove the first equivalence because $\tilde{\zeta}_A = \zeta_{-A}$. Let $\mathfrak{I}$ be a hermitian ideal of $\mathbb{B}_H$. The identities

$$T_1T_2 - S_1S_2 = (T_1 - S_1)\,T_2 + S_1(T_2 - S_2), \quad T^{-1} - S^{-1} = -T^{-1}(T-S)S^{-1}$$

prove that

$$\begin{aligned} &T_i, S_i \in \mathbb{B}_H, \;\; T_i - S_i \in \mathfrak{I} \;\;\Rightarrow\;\; T_1S_1 - T_2S_2 \in \mathfrak{I}, \\ &T^{-1}, S^{-1} \in \mathbb{B}_H, \;\; T - S \in \mathfrak{I} \;\;\Rightarrow\;\; T^{-1} - S^{-1} \in \mathfrak{I}. \end{aligned}$$

Suppose now $A, B \in \mathbb{D}_H$ normal such that $AB = BA$ and $A - B \in \mathfrak{I}$. Then $\eta_A^2 - \eta_B^2 = BB^\dagger - AA^\dagger \in \mathfrak{I}$. The functional calculus for normal operators shows that $\eta_A\eta_B = \eta_B\eta_A$, so $\eta_A - \eta_B = (\eta_A + \eta_B)^{-1}(\eta_A^2 - \eta_B^2) \in \mathfrak{I}$ since $(\eta_A + \eta_B)^{-1} \in \mathbb{B}_H$ because η_A and η_B are positive with bounded inverse. If, besides, $T, S \in \overline{\mathbb{D}}_H$ are such that $T - S \in \mathfrak{I}$, then $\varpi_A(T) - \varpi_B(S) = SB^\dagger - TA^\dagger \in \mathfrak{I}$ and $\varpi_A^*(T) - \varpi_B^*(S) = T - S + B - A \in \mathfrak{I}$. In consequence, $T - S \in \mathfrak{I} \Rightarrow \zeta_A(T) - \zeta_B(S) \in \mathfrak{I}$. Substituting in this result A, B by $-A, -B$ and T, S by $\zeta_A(T), \zeta_B(S)$ respectively, we also find the opposite inclusion.

$\square$

Taking into account that $\mathcal{A}(\boldsymbol{\alpha})$ is diagonal, the above result has the following immediate consequence.

Corollary 6.2. *If $\mathfrak{I}$ is a hermitian ideal of $\mathbb{B}_\ell^2$ and $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are sequences compactly included in $\mathbb{D}$, the condition $\mathcal{A}(\boldsymbol{\alpha}) - \mathcal{A}(\boldsymbol{\beta}) \in \mathfrak{I}$ implies the equivalence*

$$\mathcal{C}(\boldsymbol{a}) - \mathcal{C}(\boldsymbol{b}) \in \mathfrak{I} \;\Leftrightarrow\; \mathcal{U}(\boldsymbol{a}, \boldsymbol{\alpha}) - \mathcal{U}(\boldsymbol{b}, \boldsymbol{\beta}) \in \mathfrak{I}$$

for any sequences $\boldsymbol{a}, \boldsymbol{b}$ in $\overline{\mathbb{D}}$.

Besides, from the factorization $\mathcal{C}(\boldsymbol{a}) = \mathcal{C}_o(\boldsymbol{a})\,\mathcal{C}_e(\boldsymbol{a})$ we find that, for any ideal $\mathfrak{I}$ of $\mathbb{B}_{\ell^2}$,

$$\mathcal{C}_o(\boldsymbol{a}) - \mathcal{C}_o(\boldsymbol{b}),\; \mathcal{C}_e(\boldsymbol{a}) - \mathcal{C}_e(\boldsymbol{b}) \in \mathfrak{I} \;\Rightarrow\; \mathcal{C}(\boldsymbol{a}) - \mathcal{C}(\boldsymbol{b}) \in \mathfrak{I}.$$

In fact, many perturbation results for the five diagonal matrix $\mathcal{C}(\boldsymbol{a})$ are known due to the extensive use of this matrix during the last years for the spectral analysis of OP on $\mathbb{T}$.

The perturbation results that we will use are the invariance of the essential spectrum for normal operators related by a compact perturbation (Weyl's theorem: see [35] and [6, 29]), and the invariance of the absolutely continuous spectrum for unitary operators related by a trace class perturbation (Kato-Birman theorem: see [22, 7] and [8]). Given an operator T, its essential and absolutely continuous spectrum will be denoted $\sigma_e(T)$ and $\sigma_{ac}(T)$ respectively. In the case of a normal operator, $\sigma_e(T)$ is constituted by the limit points of $\sigma(T)$ and the eigenvalues with infinite geometric multiplicity. In particular, for any measure μ on $\mathbb{T}$, $\sigma_e(T_\mu)$ is the set $\{\mathrm{supp}\mu\}'$ of limit points of $\mathrm{supp}\mu$ and $\sigma_{ac}(T_\mu)$ is the support of the absolutely continuous part μ_{ac} of μ.

There are several ways to characterize the compactness of an operator but, in the case of an operator represented by a band matrix, a very practical characterization is available: compactness is equivalent to stating that all the diagonals converge to zero. If the matrix is not banded the convergence of the diagonals to zero is only a necessary condition for the compactness. The compactness can be also used to characterize certain properties of the essential spectrum. For instance, given a unitary operator T, $\sigma_e(T) \subset \{\lambda_1, \dots, \lambda_n\}$ iff $(\lambda_1 - T) \cdots (\lambda_n - T)$ is compact (Krein's theorem: see [2] and [16]).

The trace class operators, i.e., the operators T such that $\sqrt{T^\dagger T}$ has finite trace, are more difficult to characterize, even if they are represented by a band matrix. Nevertheless, any infinite matrix $(k_{i,j})$ that satisfies the condition $\sum_{i,j} |k_{i,j}| < \infty$ represents a trace class operator on ℓ^2.

Concerning the compactness and trace class character of $\mathcal{U}(\boldsymbol{a}, \boldsymbol{\alpha}) - \mathcal{U}(\boldsymbol{b}, \boldsymbol{\beta})$, Corollary 6.2 implies that it is a consequence of the same property for $\mathcal{A}(\boldsymbol{\alpha}) - \mathcal{A}(\boldsymbol{\beta})$ and $\mathcal{C}(\boldsymbol{a}) - \mathcal{C}(\boldsymbol{b})$. The diagonal matrix $\mathcal{A}(\boldsymbol{\alpha}) - \mathcal{A}(\boldsymbol{\beta})$ represents a compact operator iff $\lim_n (\alpha_n - \beta_n) = 0$, and is trace class iff $\boldsymbol{\alpha} - \boldsymbol{\beta} \in \ell^2$. As for the compactness and trace class arguments for $\mathcal{C}(\boldsymbol{a}) - \mathcal{C}(\boldsymbol{b})$ in the applications that we will discuss, they follow the same lines as in [30].

As a first group of applications in the study of the dependence $\mu(\boldsymbol{a}, \boldsymbol{\alpha})$, we will analyze the extreme behaviors corresponding to a sequence $\boldsymbol{a}$ converging to zero or (subsequently) to the unit circle. In what follows $\mathrm{Lim}_n x_n$ means the set of limit points of a sequence (x_n) in $\mathbb{C}$.

Theorem 6.3. *If $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$, then:*

1. $\lim\limits_n a_n = 0 \;\Rightarrow\; \mathrm{supp}\mu(\boldsymbol{a}, \boldsymbol{\alpha}) = \mathbb{T}$.
2. $\lim\limits_n |a_n| = 1 \;\Rightarrow\; \{\mathrm{supp}\mu(\boldsymbol{a}, \boldsymbol{\alpha})\}' = \mathop{\mathrm{Lim}}\limits_n \tilde{\zeta}_n(-\overline{a}_n a_{n+1})$.
3. $\limsup\limits_n |a_n| = 1 \;\Rightarrow\; \mu(\boldsymbol{a}, \boldsymbol{\alpha})$ *singular.*

Proof. It is straightforward to check that, for any sequence $\boldsymbol{\alpha}$ in $\mathbb{D}$, the ORF $(\phi_n)_{n \geq 0}$ corresponding to the Lebesgue measure

$$dm(e^{i\theta}) = \frac{d\theta}{2\pi} = \frac{1}{2\pi i} \frac{dz}{z}, \qquad z = e^{i\theta}, \qquad \theta \in [0, 2\pi),$$

are given by

$$\begin{aligned} &\phi_0 = 1, \\ &\phi_n = \eta_n \frac{\varpi_0^*}{\varpi_n} B_{n-1}, \qquad n \geq 1, \end{aligned}$$

and satisfy recurrence (10) with parameters $a_n = 0$. Therefore, when $\boldsymbol{\alpha}$ is compactly supported in $\mathbb{D}$, the unitary matrix $\mathcal{U}(0, \boldsymbol{\alpha})$ represents the multiplication operator T_m. So, $\sigma(\mathcal{U}(0, \boldsymbol{\alpha})) = \operatorname{supp} m = \mathbb{T}$.

Now, suppose an arbitrary sequence $\boldsymbol{a}$ in $\mathbb{D}$ such that $\lim_n a_n = 0$. Then $\mathcal{C}(\boldsymbol{a}) - \mathcal{C}(0)$ is compact, thus $\mathcal{U}(\boldsymbol{a}, \boldsymbol{\alpha}) - \mathcal{U}(0, \boldsymbol{\alpha})$ is compact too. Hence, Weyl's theorem implies $\{\operatorname{supp} \mu(\boldsymbol{a}, \boldsymbol{\alpha})\}' = \{\operatorname{supp} m\}' = \mathbb{T}$, that is, $\operatorname{supp} \mu(\boldsymbol{a}, \boldsymbol{\alpha}) = \mathbb{T}$.

If $\lim_n |a_n| = 1$, then $\mathcal{C}(\boldsymbol{a}) - \mathcal{D}(\boldsymbol{a})$ is compact, where $\mathcal{D}(\boldsymbol{a})$ is the diagonal matrix

$$\mathcal{D}(\boldsymbol{a}) = \begin{pmatrix} -a_1 & & & \\ & -\overline{a}_1 a_2 & & \\ & & -\overline{a}_2 a_3 & \\ & & & \ddots \end{pmatrix}.$$

Therefore, Proposition 6.1 implies that $\mathcal{U}(\boldsymbol{a}, \boldsymbol{\alpha}) - \tilde{\zeta}_{\mathcal{A}(\boldsymbol{\alpha})}(\mathcal{D}(\boldsymbol{a}))$ is compact too. Notice that

$$\tilde{\zeta}_{\mathcal{A}(\boldsymbol{\alpha})}(\mathcal{D}(\boldsymbol{a})) = \begin{pmatrix} \tilde{\zeta}_0(-a_1) & & & \\ & \tilde{\zeta}_1(-\overline{a}_1 a_2) & & \\ & & \tilde{\zeta}_2(-\overline{a}_2 a_3) & \\ & & & \ddots \end{pmatrix}$$

is diagonal and bounded, so it is normal and Weyl's theorem states that $\{\operatorname{supp} \mu(\boldsymbol{a}, \boldsymbol{\alpha})\}' = \operatorname{Lim}_n \tilde{\zeta}_n(-\overline{a}_n a_{n+1})$.

Finally, assume that $\limsup_n |a_n| = 1$. This means that there is a subsequence $(a_n)_{n \in \mathcal{I}}$, $\mathcal{I} \subset \mathbb{N}$, such that $\lim_{n \in \mathcal{I}} a_n = a \in \mathbb{T}$. Without loss of generality we can suppose $\sum_{n \in \mathcal{I}} |a_n - a|^{1/2} < \infty$, so that $\sum_{n \in \mathcal{I}} (|a_n - a| + \rho_n) < \infty$ because $\rho_n \leq \sqrt{2|a_n - a|}$. Let $\boldsymbol{b}$ be the sequence defined by

$$b_n = \begin{cases} a & \text{if } n \in \mathcal{I}, \\ a_n & \text{if } n \notin \mathcal{I}. \end{cases}$$

The condition $\sum_{n \in \mathcal{I}} (|a - a_n| + \rho_n) < \infty$ ensures that $\mathcal{C}_o(\boldsymbol{a}) - \mathcal{C}_o(\boldsymbol{b})$ and $\mathcal{C}_e(\boldsymbol{a}) - \mathcal{C}_e(\boldsymbol{b})$ are trace class, so the same holds for $\mathcal{U}(\boldsymbol{a}, \boldsymbol{\alpha}) - \mathcal{U}(\boldsymbol{b}, \boldsymbol{\alpha})$. The Birman-Krein theorem states that $\operatorname{supp} \mu_{ac}(\boldsymbol{a}, \boldsymbol{\alpha}) = \sigma_{ac}(\mathcal{U}(\boldsymbol{b}, \boldsymbol{\alpha}))$, but the fact that $b_n \in \mathbb{T}$ for infinitely many values of n implies that $\mathcal{U}(\boldsymbol{b}, \boldsymbol{\alpha})$ decomposes as a direct sum of finite matrices, so it has a pure point spectrum and, hence, it has no absolutely continuous part. Therefore, $\mu_{ac}(\boldsymbol{a}, \boldsymbol{\alpha}) = 0$.

$\square$

We can also obtain general conditions for the invariance of $\{\mathrm{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}'$ and $\mathrm{supp}\mu_{ac}(\boldsymbol{a},\boldsymbol{\alpha})$.

Theorem 6.4. *If $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$, then:*

1. $\lim\limits_n(\alpha_n-\beta_n)=\lim\limits_n(a_n-b_n)=0 \;\Rightarrow\; \{\mathrm{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}'=\{\mathrm{supp}\mu(\boldsymbol{b},\boldsymbol{\beta})\}'$.

2. $\sum\limits_n(|\alpha_n-\beta_n|+|a_n-b_n|)<\infty \;\Rightarrow\; \mathrm{supp}\mu_{ac}(\boldsymbol{a},\boldsymbol{\alpha})=\mathrm{supp}\mu_{ac}(\boldsymbol{b},\boldsymbol{\beta})$.

3. *If $b_n=\lambda_n a_n$ with $\lambda_n\in\mathbb{C}$, then:*

$$\lim_n|\lambda_n|=\lim_n\lambda_{n+1}\overline{\lambda}_n=1 \;\Rightarrow\; \{\mathrm{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}'=\{\mathrm{supp}\mu(\boldsymbol{b},\boldsymbol{\alpha})\}',$$
$$\sum_n(||\lambda_n|^2-1|+|\lambda_{n+1}\overline{\lambda}_n-1|)<\infty \;\Rightarrow\; \mathrm{supp}\mu_{ac}(\boldsymbol{a},\boldsymbol{\alpha})=\mathrm{supp}\mu_{ac}(\boldsymbol{b},\boldsymbol{\alpha}).$$

4. $\beta_n=\alpha_{n+N},\; b_n=a_{n+N} \;\Rightarrow\; \begin{cases}\{\mathrm{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}'=\{\mathrm{supp}\mu(\boldsymbol{b},\boldsymbol{\beta})\}',\\ \mathrm{supp}\mu_{ac}(\boldsymbol{a},\boldsymbol{\alpha})=\mathrm{supp}\mu_{ac}(\boldsymbol{b},\boldsymbol{\beta}).\end{cases}$

Proof. First, notice that any of the hypothesis of the theorem ensure that $\boldsymbol{\beta}$ is compactly included in $\mathbb{D}$ when $\boldsymbol{\alpha}$ satisfies the same property. Thus, the spectral theory that we have developed works for both sequences, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Concerning the notation, in what follows we write $\rho_n=\sqrt{1-|a_n|^2}$, as usually, and $\sigma_n=\sqrt{1-|b_n|^2}$.

To prove the first item, notice that the inequality

$$|\rho_n-\sigma_n|^2\leq|\rho_n^2-\sigma_n^2|=||a_n|^2-|b_n|^2|\leq 2|a_n-b_n|$$

implies that the conditions $\lim_n(\alpha_n-\beta_n)=\lim_n(a_n-b_n)=0$ ensure the compactness of $\mathcal{A}(\boldsymbol{\alpha})-\mathcal{A}(\boldsymbol{\beta})$, $\mathcal{C}_o(\boldsymbol{a})-\mathcal{C}_o(\boldsymbol{b})$ and $\mathcal{C}_e(\boldsymbol{a})-\mathcal{C}_e(\boldsymbol{b})$. In consequence, $\mathcal{U}(\boldsymbol{a},\boldsymbol{\alpha})-\mathcal{U}(\boldsymbol{b},\boldsymbol{\beta})$ is compact too and Weyl's theorem implies the equality $\{\mathrm{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}'=\{\mathrm{supp}\mu(\boldsymbol{b},\boldsymbol{\beta})\}'$.

Suppose now $\sum_n(|\alpha_n-\beta_n|+|a_n-b_n|)<\infty$. If $\limsup_n|a_n|=1$, then $\limsup_n|b_n|=1$, so we conclude from Theorem 6.3.3 that $\mathrm{supp}\mu_{ac}(\boldsymbol{a},\boldsymbol{\alpha})=\mathrm{supp}\mu_{ac}(\boldsymbol{b},\boldsymbol{\beta})=\emptyset$. If, on the contrary, $\limsup_n|a_n|<1$, then $|a_n|,|b_n|\leq r$ for some $r<1$. Taking into account the inequality

$$|\rho_n-\sigma_n|\leq\frac{||a_n|^2-|b_n|^2|}{\rho_n+\sigma_n}\leq\frac{r}{\sqrt{1-r^2}}|a_n-b_n|,$$

$\sum_n (|\alpha_n - \beta_n| + |a_n - b_n|) < \infty$ implies that $\mathcal{A}(\boldsymbol{\alpha}) - \mathcal{A}(\boldsymbol{\beta})$, $\mathcal{C}_o(\boldsymbol{a}) - \mathcal{C}_o(\boldsymbol{b})$ and $\mathcal{C}_e(\boldsymbol{a}) - \mathcal{C}_e(\boldsymbol{b})$ are trace class. Thus, $\mathcal{U}(\boldsymbol{a}, \boldsymbol{\alpha}) - \mathcal{U}(\boldsymbol{b}, \boldsymbol{\beta})$ is trace class too and, from the Birman-Krein theorem, $\mathrm{supp}\mu_{ac}(\boldsymbol{a}, \boldsymbol{\alpha}) = \mathrm{supp}\mu_{ac}(\boldsymbol{b}, \boldsymbol{\beta})$.

Consider $b_n = \lambda_n a_n$ with $\lim_n |\lambda_n| = \lim_n \lambda_{n+1}\overline{\lambda}_n = 1$. We can write $\lambda_n = |\lambda_n| e^{i\theta_n}$ with $\theta_n \in [\theta_{n-1} - \pi, \theta_{n-1} + \pi)$, so that $\lim_n |\theta_{n+1} - \theta_n| = 0$. Define

$$U = \begin{pmatrix} u_1 & & & & \\ & \overline{u}_2 & & & \\ & & u_3 & & \\ & & & \overline{u}_4 & \\ & & & & \ddots \end{pmatrix}, \qquad u_n = e^{i\theta_n/2}.$$

The identity

$$\begin{pmatrix} u_n & \\ & \overline{u}_{n+1} \end{pmatrix} \Theta_n(\boldsymbol{a}) \begin{pmatrix} u_n & \\ & \overline{u}_{n+1} \end{pmatrix} - \Theta_n(\boldsymbol{b}) =$$
$$= \begin{pmatrix} a_n u_n^2 (|\lambda_n|^2 - 1) & \rho_n \overline{u}_{n+1} u_n - \sigma_n \\ \rho_n \overline{u}_{n+1} u_n - \sigma_n & \overline{a}_n \overline{u}_n^2 (\overline{u}_{n+1}^2 u_n^2 - |\lambda_n|^2) \end{pmatrix},$$

together with $\lim_n u_{n+1}\overline{u}_n = 1$ and $|\rho_n - \sigma_n|^2 \leq |\rho_n^2 - \sigma_n^2| = |1 - |\lambda_n|^2||a_n|^2$, shows that $U\mathcal{C}_o(\boldsymbol{a})U - \mathcal{C}_o(\boldsymbol{b})$ and $U^\dagger \mathcal{C}_e(\boldsymbol{a}) U^\dagger - \mathcal{C}_e(\boldsymbol{b})$ are compact. This implies the compactness of $U\mathcal{C}(\boldsymbol{a})U^\dagger - \mathcal{C}(\boldsymbol{b}) = U\mathcal{C}_o(\boldsymbol{a})U\, U^\dagger \mathcal{C}_e(\boldsymbol{a}) U^\dagger - \mathcal{C}_o(\boldsymbol{b})\, \mathcal{C}_e(\boldsymbol{b})$, which, bearing in mind Proposition 6.1, is equivalent to the compactness of $U\mathcal{U}(\boldsymbol{a}, \boldsymbol{\alpha})U^\dagger - \mathcal{U}(\boldsymbol{b}, \boldsymbol{\alpha}) = \zeta_{\mathcal{A}(\boldsymbol{\alpha})}(U\mathcal{C}(\boldsymbol{a})U^\dagger) - \zeta_{\mathcal{A}(\boldsymbol{\alpha})}(\mathcal{C}(\boldsymbol{b}))$. Therefore, Weyl's Theorem implies that $\{\mathrm{supp}\mu(\boldsymbol{a}, \boldsymbol{\alpha})\}' = \{\mathrm{supp}\mu(\boldsymbol{b}, \boldsymbol{\alpha})\}'$.

When $\sum_n (||\lambda_n|^2 - 1| + |\lambda_{n+1}\overline{\lambda}_n - 1|) < \infty$ we have to consider again two possibilities. If $\limsup_n |a_n| = 1$, necessarily $\limsup_n |b_n| = 1$ and $\mathrm{supp}\mu_{ac}(\boldsymbol{a}, \boldsymbol{\alpha}) = \mathrm{supp}\mu_{ac}(\boldsymbol{b}, \boldsymbol{\alpha}) = \emptyset$ from Theorem 6.3.3. If $\limsup_n |a_n| < 1$, then $|a_n|, |b_n| \leq r$ for some $r < 1$, so the relations

$$\begin{aligned}
&|\rho_n - \sigma_n| \leq \frac{||a_n|^2 - |b_n|^2|}{\rho_n + \sigma_n} \leq \frac{r^2}{2\sqrt{1 - r^2}} |1 - |\lambda_n|^2|, \\
&|u_{n+1}^2 \overline{u}_n^2 - 1| \sim |\lambda_{n+1}\overline{\lambda}_n - |\lambda_{n+1}\overline{\lambda}_n|| \leq 2|\lambda_{n+1}\overline{\lambda}_n - 1|, \\
&|u_{n+1}\overline{u}_n - 1| = \frac{|u_{n+1}^2 \overline{u}_n^2 - 1|}{|u_{n+1}\overline{u}_n + 1|} \sim \frac{1}{2} |u_{n+1}^2 \overline{u}_n^2 - 1| \leq |\lambda_{n+1}\overline{\lambda}_n - 1|,
\end{aligned}$$

ensure that $U\mathcal{C}_o(\boldsymbol{a})U - \mathcal{C}_o(\boldsymbol{b})$ and $U^\dagger \mathcal{C}_e(\boldsymbol{a}) U^\dagger - \mathcal{C}_e(\boldsymbol{b})$ are trace class. The Birman-Krein theorem then proves that $\mathrm{supp}\mu_{ac}(\boldsymbol{a}, \boldsymbol{\alpha}) = \mathrm{supp}\mu_{ac}(\boldsymbol{b}, \boldsymbol{\alpha})$ similarly to the previous case.

Finally, let $b_n = a_{n+N}$ and $\beta_n = \alpha_{n+N}$ for some $N \in \mathbb{N}$. Consider the sequences $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{\alpha}}$ given by

$$\tilde{a}_n = \begin{cases} 1 & \text{if } n \leq N, \\ a_n & \text{if } n > N, \end{cases} \qquad \tilde{\alpha}_n = \begin{cases} 0 & \text{if } n \leq N, \\ \alpha_n & \text{if } n > N. \end{cases}$$

$\mathcal{A}(\boldsymbol{\alpha}) - \mathcal{A}(\tilde{\boldsymbol{\alpha}})$, $\mathcal{C}_o(\boldsymbol{a}) - \mathcal{C}_o(\tilde{\boldsymbol{a}})$ and $\mathcal{C}_e(\boldsymbol{a}) - \mathcal{C}_e(\tilde{\boldsymbol{a}})$ are finite rank, therefore $\mathcal{U}(\boldsymbol{a},\boldsymbol{\alpha}) - \mathcal{U}(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{\alpha}})$ is compact and trace class. Besides, we have the decomposition $\mathcal{U}(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{\alpha}}) = -I_N \oplus \mathcal{U}(\boldsymbol{b},\boldsymbol{\beta})$, so $\mathcal{U}(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{\alpha}})$ and $\mathcal{U}(\boldsymbol{b},\boldsymbol{\beta})$ have the same essential and absolutely continuous spectrum. As a consequence of these facts, the Weyl and Birman-Krein theorems give $\{\text{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}' = \{\text{supp}\mu(\boldsymbol{b},\boldsymbol{\beta})\}'$ and $\text{supp}\mu_{ac}(\boldsymbol{a},\boldsymbol{\alpha}) = \text{supp}\mu_{ac}(\boldsymbol{b},\boldsymbol{\beta})$.

□

Combining the different results of the previous theorem we can obtain a more general one.

Theorem 6.5. *For any sequence $\boldsymbol{\alpha}$ compactly included in $\mathbb{D}$:*

1. *If* $\lim_n(\alpha_{n+N} - \beta_n) = \lim_n(\lambda_n a_{n+N} - b_n) = 0$, $\lim_n |\lambda_n| = \lim_n \lambda_{n+1}\overline{\lambda}_n = 1$, *then* $\{\text{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}' = \{\text{supp}\mu(\boldsymbol{b},\boldsymbol{\beta})\}'$.
2. *If* $\sum_n (|\alpha_{n+N} - \beta_n| + |\lambda_n a_{n+N} - b_n| + ||\lambda_n|^2 - 1| + |\lambda_{n+1}\overline{\lambda}_n - 1|) < \infty$, *then* $\text{supp}\mu_{ac}(\boldsymbol{a},\boldsymbol{\alpha}) = \text{supp}\mu_{ac}(\boldsymbol{b},\boldsymbol{\beta})$.

A particular case of this theorem is worthwhile to be emphasized.

Corollary 6.6. *Let $\alpha \in \mathbb{D}$, $a \in [0,1]$, $\lambda \in \mathbb{T}$ and*

$$\Gamma_{\lambda,a} = \{\lambda e^{i\theta} : |\theta| < 2\arcsin a\}.$$

1. *If* $\lim_n \alpha_n = \alpha$, $\lim_n |a_n| = a$ *and* $\lim_n \frac{a_{n+1}}{a_n} = \lambda$, *then*
$$\{\text{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}' = \mathbb{T} \setminus \tilde{\zeta}_\alpha(\Gamma_{\lambda,a}).$$
2. *If* $\sum_n \left(|\alpha_n - \alpha| + ||a_n| - a| + \left|\frac{a_{n+1}}{a_n} - \lambda\right|\right) < \infty$, *then*
$$\text{supp}\mu_{ac}(\boldsymbol{a},\boldsymbol{\alpha}) = \mathbb{T} \setminus \tilde{\zeta}_\alpha(\Gamma_{\lambda,a}).$$

Proof. Let us write $a_n = |a_n| v_n$ with $v_n \in \mathbb{T}$. Notice that $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$ because it is convergent in $\mathbb{D}$. Therefore, we can apply Theorem 6.5 to $\mu(\boldsymbol{a}, \boldsymbol{\alpha})$ and $\mu(\boldsymbol{b}, \boldsymbol{\beta})$ with $\beta_n = \alpha$, $b_n = \lambda^n a$ and $\lambda_n = \lambda^n \overline{v}_n$. Taking into account the relation

$$|\lambda_{n+1}\overline{\lambda}_n - 1| = \left|\lambda - \frac{v_{n+1}}{v_n}\right| \leq \left|\lambda - \frac{a_{n+1}}{a_n}\right| + \left|\frac{a_{n+1}}{a_n} - \frac{v_{n+1}}{v_n}\right| =$$
$$= \left|\lambda - \frac{a_{n+1}}{a_n}\right| + \left|\frac{|a_{n+1}|}{|a_n|} - 1\right| \leq 2\left|\frac{a_{n+1}}{a_n} - \lambda\right|,$$

we find that $\{\text{supp}\mu(\boldsymbol{a}, \boldsymbol{\alpha})\}' = \{\text{supp}\mu(\boldsymbol{b}, \boldsymbol{\beta})\}'$ under the assumptions of item 1, and $\text{supp}\mu_{ac}(\boldsymbol{a}, \boldsymbol{\alpha}) = \text{supp}\mu_{ac}(\boldsymbol{b}, \boldsymbol{\beta})$ under the hypothesis of item 2. On the other hand, from the comments at the beginning of Section 4, we know that $\mu(\boldsymbol{b}, \boldsymbol{\beta}) = \nu_\alpha$, where $\nu = \mu(\boldsymbol{b}, 0)$ is the measure on $\mathbb{T}$ whose OP have parameters $\lambda^n a$ and ν_α is defined by $\nu_\alpha(\Delta) = \nu(\zeta_\alpha(\Delta))$ for any Borel subset Δ of $\mathbb{T}$. Therefore, $\{\text{supp}\nu_\alpha\}' = \tilde{\zeta}_\alpha(\{\text{supp}\nu\}')$, $\text{supp}(\nu_\alpha)_{ac} = \tilde{\zeta}_\alpha(\text{supp}\nu_{ac})$ and the corollary follows from the well known result $\{\text{supp}\nu\}' = \text{supp}\nu_{ac} = \mathbb{T} \setminus \Gamma_{\lambda,a}$.

□

If $a = 0$, Corollary 6.6.1 is a direct consequence of Theorem 6.3.1, while Corollary 6.6.2 can be derived from Szegő's Theorem for OP on $\mathbb{T}$: Theorem 6.4.2 implies that $\text{supp}\mu_{ac}(\boldsymbol{a}, \boldsymbol{\alpha}) = \text{supp}\mu_{ac}(\boldsymbol{a}, \boldsymbol{\beta})$ for $\beta_n = \alpha$ whenever $\sum_n |\alpha_n - \alpha| < \infty$. $\mu(\boldsymbol{a}, \boldsymbol{\beta}) = \nu_\alpha$, where now $\nu = \mu(\boldsymbol{a}, 0)$, and the condition $\sum_n |a_n| < \infty$ gives $\text{supp}\nu_{ac} = \mathbb{T}$ because ν is in the Szegő class $\boldsymbol{a} = \mathcal{S}_0(\nu) \in \ell^2$. Hence, $\text{supp}\mu_{ac}(\boldsymbol{a}, \boldsymbol{\alpha}) = \text{supp}(\nu_\alpha)_{ac} = \mathbb{T}$. In fact, this reasoning proves that the equality $\text{supp}\mu_{ac}(\boldsymbol{a}, \boldsymbol{\alpha}) = \mathbb{T}$ holds under the more general condition $\sum_n (|\alpha_n - \alpha| + |a_n|^2) < \infty$.

Corollary 6.6 of Theorem 6.5 can be understood also as an example of the following general result. It says that, when $\boldsymbol{\alpha}$ is convergent in $\mathbb{D}$, the analysis of $\{\text{supp}\mu(\boldsymbol{a}, \boldsymbol{\alpha})\}'$ and $\text{supp}\mu_{ac}(\boldsymbol{a}, \boldsymbol{\alpha})$ can be related to the much more known polynomial case, corresponding to $\boldsymbol{\alpha} = 0$.

Theorem 6.7. *Let $\alpha \in \mathbb{D}$.*

1. $\lim_n \alpha_n = \alpha \;\Rightarrow\; \{\text{supp}\mu(\boldsymbol{a}, \boldsymbol{\alpha})\}' = \tilde{\zeta}_\alpha(\{\text{supp}\mu(\boldsymbol{a}, 0)\}')$.

2. $\sum_n |\alpha_n - \alpha| < \infty \;\Rightarrow\; \text{supp}\mu_{ac}(\boldsymbol{a}, \boldsymbol{\alpha}) = \tilde{\zeta}_\alpha(\text{supp}\mu_{ac}(\boldsymbol{a}, 0))$.

Proof. Again $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$ because it is convergent in $\mathbb{D}$. So, if $\beta_n = \alpha$, Theorem 6.4 implies that $\{\mathrm{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}' = \{\mathrm{supp}\mu(\boldsymbol{a},\boldsymbol{\beta})\}'$ when $\lim_n \alpha_n = \alpha$, and $\mathrm{supp}\mu_{ac}(\boldsymbol{a},\boldsymbol{\alpha}) = \mathrm{supp}\mu_{ac}(\boldsymbol{a},\boldsymbol{\beta})$ when $\sum_n |\alpha_n - \alpha| < \infty$. On the other hand, $\mu(\boldsymbol{a},\boldsymbol{\beta}) = \nu_\alpha$ with $\nu = \mu(\boldsymbol{a},0)$. As in the proof of Corollary 6.6, the result follows from the relation between ν and ν_α.

□

The importance of the above theorem is due to the numerous known results for the relation between μ and $\boldsymbol{a}$ in the case of OP on $\mathbb{T}$. Theorem 6.7 permits us to translate some of these results to those ORF on $\mathbb{T}$ whose poles converge in $\mathbb{E}$. For instance, Corollary 6.6.1 can be understood as the translation to this kind of ORF of a result for OP on $\mathbb{T}$ due to Barrios-López (see [5]). This result was generalized later on in [26] as an improvement of a partial extension appearing in [14]. The corresponding translation of this generalization to ORF states that Corollary 6.6.1 holds even if we substitute the condition $\lim_n |a_n| = a$ by the more general one $\liminf_n |a_n| = a$.

All the above results provide only sufficient conditions on the sequences $\boldsymbol{\alpha}$ and $\boldsymbol{a}$ to ensure a certain property for the measure $\mu(\boldsymbol{a},\boldsymbol{\alpha})$. On the contrary, Krein's theorem permits us to characterize exactly those measures $\mu(\boldsymbol{a},\boldsymbol{\alpha})$ with a fixed finite set $\{\mathrm{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}'$. The characterization is in terms of the compactness of a matrix depending on $\boldsymbol{a}$ and $\boldsymbol{\alpha}$. The fact that, contrary to the polynomial case, this matrix is not banded makes difficult to translate its compactness into equivalent conditions for the sequences $\boldsymbol{a}$ and $\boldsymbol{\alpha}$. Nevertheless, in the case of $\{\mathrm{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}'$ with at most two points we can find explicitly such equivalent conditions.

Theorem 6.8. *If $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{T}$, then:*

1. $\{\mathrm{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}' = \{\lambda\}$ *iff*
$$\lim_n \tilde{\zeta}_n(-\overline{a}_n a_{n+1}) = \lambda.$$

2. $\{\mathrm{supp}\mu(\boldsymbol{a},\boldsymbol{\alpha})\}' \subset \{\lambda_1, \lambda_2\}$ *iff*
$$\begin{aligned}
&\lim{}_n \rho_n \rho_{n+1} = 0,\\
&\lim_n \rho_n \left(\frac{\varpi_n(\lambda_1)}{\varpi_n(\alpha_n)} k_n(\lambda_2) - \frac{\varpi^*_{n-1}(\lambda_2)}{\varpi_{n-1}(\alpha_{n-1})} k_{n-1}(\lambda_1) \right) = 0,\\
&\lim_n \Big(\overline{k_n(\lambda_1)} k_n(\lambda_2) + (\rho_n^-)^2\, \overline{\varpi^*_{n-1}(\lambda_1)}\, \varpi^*_{n-1}(\lambda_2) + \\
&\qquad\qquad + (\rho_{n+1}^+)^2\, \overline{\varpi_{n+1}(\lambda_1)}\, \varpi_{n+1}(\lambda_2) \Big) = 0,
\end{aligned}$$

where $k_n(z) = a_n \varpi_n^*(z) + a_{n+1}\varpi_n(z)$.

Proof. We are dealing only with measures μ with an infinite support on $\mathbb{T}$, thus, suppμ has at least one limit point in $\mathbb{T}$. Hence, from Krein's theorem, $\{\text{supp}\mu\}' = \{\lambda\}$ iff $\lambda - T_\mu$ is compact, i.e., iff $\lambda - \mathcal{U}$ is compact. (17) yields

$$\lambda - \mathcal{U} = \lambda - \tilde{\zeta}_{\mathcal{A}}(\mathcal{C}) = \eta_{\mathcal{A}}^{-1} \varpi_{\mathcal{A}}(\lambda)\,(\zeta_{\mathcal{A}}(\lambda) - \mathcal{C})\,\tilde{\varpi}_{\mathcal{A}}(\mathcal{C})^{-1}\eta_{\mathcal{A}}. \tag{44}$$

Bearing in mind that $\eta_{\mathcal{A}}$, $\varpi_{\mathcal{A}}(\lambda)$ and $\tilde{\varpi}_{\mathcal{A}}(\mathcal{C})$ are bounded with bounded inverse, the above expression shows that the compactness of $\lambda-\mathcal{U}$ is equivalent to the compactness of $\zeta_{\mathcal{A}}(\lambda)-\mathcal{C}$. On the other hand, $\zeta_{\mathcal{A}}(\lambda)-\mathcal{C}$ is compact iff $\lim_n \rho_n = 0$ and $\lim_n(\zeta_n(\lambda) + \overline{a}_n a_{n+1}) = 0$. However, the first of these conditions is a consequence of the second one because $|\zeta_n(\lambda) + \overline{a}_n a_{n+1}| \geq 1 - |a_n|$ since $\lambda \in \mathbb{T}$. Also, taking into account (17),

$$\varpi_n(\lambda)\,(\zeta_n(\lambda) + \overline{a}_n a_{n+1}) = (\lambda - \tilde{\zeta}_n(-\overline{a}_n a_{n+1}))\,\tilde{\varpi}_n(-\overline{a}_n a_{n+1}).$$

Therefore, $\lim_n(\zeta_n(\lambda) + \overline{a}_n a_{n+1}) = 0$ iff $\lim_n(\lambda - \tilde{\zeta}_n(-\overline{a}_n a_{n+1})) = 0$ because $2 > |\varpi_n(\lambda)|, |\tilde{\varpi}_n(-\overline{a}_n a_{n+1})| \geq 1 - |\alpha_n|$ and $\boldsymbol{\alpha}$ is compactly supported in $\mathbb{D}$.

As for the case of two limit points, Krein's theorem implies that the inclusion $\{\text{supp}\mu\}' \subset \{\lambda_1, \lambda_2\}$ is equivalent to the compactness of the matrix $(\lambda_1 - \mathcal{U})(\lambda_2 - \mathcal{U})$. To express this condition as the compactness of a band matrix we use (44) for the factor $\lambda_2 - \mathcal{U}$, but for $\lambda_1 - \mathcal{U}$ we use the equality

$$\lambda - \mathcal{U} = \lambda - \zeta_{-\mathcal{A}}(\mathcal{C}) = \eta_{\mathcal{A}} \varpi_{-\mathcal{A}}(\mathcal{C})^{-1}(\zeta_{\mathcal{A}}(\lambda) - \mathcal{C})\,\tilde{\varpi}_{-\mathcal{A}}(\lambda)\,\eta_{\mathcal{A}}^{-1}, \tag{45}$$

obtained from (17) and the identity $\tilde{\zeta}_{\mathcal{A}} = \zeta_{-\mathcal{A}}$. Then, similarly to the case of one limit point, we find that $(\lambda_1 - \mathcal{U})(\lambda_2 - \mathcal{U})$ is compact iff the 9-diagonal matrix $(\zeta_{\mathcal{A}}(\lambda_1) - \mathcal{C})\,\varpi_{\mathcal{A}}(\lambda_1)\,\varpi_{\mathcal{A}}(\mathcal{A})^{-1}\varpi_{\mathcal{A}}(\lambda_2)(\zeta_{\mathcal{A}}(\lambda_2) - \mathcal{C})$ is compact. This compactness condition can be equivalently formulated using a simpler band matrix obtained multiplying the above one on the left and the right by the unitary matrices $\mathcal{C}_o^\dagger$ and $\mathcal{C}_e^\dagger$ respectively. Taking into account the identity $\varpi_{\mathcal{A}}^*(z) = z\,\varpi_{\mathcal{A}}(z)^\dagger$, $z \in \mathbb{T}$, we find in this way that $\{\text{supp}\mu\}' \subset \{\lambda_1, \lambda_2\}$ iff the five-diagonal matrix $K(\lambda_1)^\dagger \varpi_{\mathcal{A}}(\mathcal{A})^{-1} K(\lambda_2)$ is compact, where $K(z) = \varpi_{\mathcal{A}}^*(z)\,\mathcal{C}_e^\dagger - \varpi_{\mathcal{A}}(z)\,\mathcal{C}_o$. Now, it is just a matter of calculating the diagonals of $K(\lambda_1)^\dagger \varpi_{\mathcal{A}}(\mathcal{A})^{-1} K(\lambda_2)$ to obtain the conditions given in the theorem.

$\square$

The implication $\lim_n \tilde{\zeta}_n(-\overline{a}_n a_{n+1}) = \lambda \in \mathbb{T} \Rightarrow \{\text{supp}\mu\}' = \{\lambda\}$ was in fact a consequence of Theorem 6.3.2. Krein's theorem adds the opposite

implication. Concerning the case of two limit points notice that, although the third condition is symmetric under the exchange of λ_1 and λ_2, the second one does not show explicitly such a symmetry. However, a detailed analysis of the second condition reveals that it is symmetric too.

It seems that there is no simple way to generalize the arguments given in the proof of Theorem 6.8 to the case of more than two limit points. The reason is that, for $n \geq 3$, identities (44) and (45) are not enough to reduce the compactness of $(\lambda_1 - \mathcal{U}) \cdots (\lambda_n - \mathcal{U})$ to the compactness of a band matrix. So, contrary to the polynomial situation (see [16] and [13, 26]), the practical application of Krein's theorem to characterize in terms of the sequences $\boldsymbol{a}$ and $\boldsymbol{\alpha}$ those measures on $\mathbb{T}$ whose support has a finite set of more than two limit points remains as an open problem in the rational case.

7 Appendix: ORF on the real line

In what follows, a measure on the real line will be probability Borel measure μ supported on an infinite subset suppμ of $\overline{\mathbb{R}}$. When ∞ is not a mass point of μ we will refer to μ as a measure on $\mathbb{R}$. Notice that we are considering all these measures as measures on $\overline{\mathbb{R}}$, no matter whether they have a mass point at ∞ or not. This means that $\infty \in$ suppμ when ∞ is a mass point of μ or when μ is a measure on $\mathbb{R}$ with unbounded standard support, so that suppμ is always closed in $\overline{\mathbb{R}}$.

Analogously to the case of the unit circle, for any measure μ on the real line it is possible to consider ORF in L^2_μ with poles in the lower half plane $\mathbb{L} = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$. For this purpose we introduce for any $\alpha \in \mathbb{U} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the linear fractional transformation

$$\zeta_\alpha(z) = \frac{\varpi^*_\alpha(z)}{\varpi_\alpha(z)}, \qquad \begin{cases} \varpi_\alpha(z) = z - \overline{\alpha}, \\ \varpi^*_\alpha(z) = z - \alpha, \end{cases}$$

which maps $\overline{\mathbb{R}}$, $\mathbb{U}$ and $\mathbb{L}$ onto $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{E}$ respectively, and has the inverse

$$\tilde{\zeta}_\alpha(z) = \frac{\tilde{\varpi}^*_\alpha(z)}{\tilde{\varpi}_\alpha(z)}, \qquad \begin{cases} \tilde{\varpi}_\alpha(z) = 1 - z, \\ \tilde{\varpi}^*_\alpha(z) = \alpha - \overline{\alpha} z. \end{cases}$$

Notice that $\varpi^*_\alpha = \varpi_{\alpha *}$, where the $*$-involution is now defined by $f_*(z) = \overline{f(\overline{z})}$, but nothing similar holds for $\tilde{\varpi}^*_\alpha$. Besides, for the distinguished value $\alpha_0 = i$, $\zeta = \zeta_{\alpha_0}$ is the Cayley transform and $\tilde{\zeta} = \tilde{\zeta}_{\alpha_0}$ its inverse.

Any sequence $\boldsymbol{\alpha} = (\alpha_n)_{n\geq 1}$ in $\mathbb{U}$ defines the products $(B_n)_{n\geq 0}$ as in (8), but with the new meaning for ζ_{α_n}. The orthonormalization in L^2_μ of $(B_n)_{n\geq 0}$ leads to a sequence $(\phi_n)_{n\geq 0}$ of ORF with respect to μ with poles in $(\overline{\alpha}_n)_{n\geq 1}$, which will be called a sequence of ORF on the real line. The study of ORF on the real line can be carried out in a completely analogous way to the case of the unit circle, so most of the results described for the last ones translate directly to the first ones with an obvious change of the meaning in the notations. In particular, the sequence $(\phi_n)_{n\geq 0}$ can be chosen such that it satisfies a recurrence like (10) depending on a sequence $\boldsymbol{a} = (a_n)_{n\geq 1}$ in $\mathbb{D}$, which establishes a surjective application $\mathcal{S}_{\boldsymbol{\alpha}}\colon \mathfrak{P} \to \mathbb{D}^\infty$, where $\mathfrak{P}$ means now the set of probability measures on $\overline{\mathbb{R}}$. This application is a bijection when $B_n(z) = \prod_{n=1}^\infty \zeta_n(z)$ diverges to zero for $z \in \mathbb{U}$, but this is equivalent now to $\sum_{n=1}^\infty \operatorname{Im}\alpha_n/(1+|\alpha_n|^2) = \infty$, which means that the poles can not approach too quickly to $\overline{\mathbb{R}}$.

Following the same strategy as in the case of the unit circle, we can develop a spectral theory for ORF on the real line. The starting point is again recurrence (10) written in the form (12), but now the positive factors η_α, $\alpha \in \mathbb{U}$, are defined by

$$\eta_\alpha = \left(\frac{\varpi_\alpha(\alpha)}{2i}\right)^{1/2} = \sqrt{\operatorname{Im}\alpha}.$$

Both, the expressions for the unit circle and the real line can be combined in $\eta_\alpha = (\varpi_\alpha(\alpha)/\varpi_{\alpha_0}(\alpha_0))^{1/2}$.

The form (12) of the recurrence is the key tool to obtain the matrix representations with respect to the ORF for the multiplication operator

$$T_\mu\colon \underset{f(z)\,\to\, zf(z)}{L^2_\mu \to L^2_\mu}$$

where μ is the corresponding orthogonality measure on the real line. If $\operatorname{supp}\mu$ is bounded, T_μ is an everywhere defined self-adjoint operator on L^2_μ. In general, T_μ is a densely defined self-adjoint operator on L^2_μ when the function z is finite μ-a.e. (see [28, page 259]), that is, when ∞ is not a mass point of μ. In this case, $\sigma_p(T_\mu) = \{\text{mass points of } \mu\}$ and $\sigma(T_\mu) = \operatorname{supp}\mu$ under the convention that $\infty \in \sigma(T_\mu)$ when T_μ has an unbounded standard spectrum. A way to deal with the case of measures with a mass point at ∞ is to work with the operator multiplication by ζ in L^2_μ, i.e.,

$$S_\mu\colon \underset{f(z)\,\to\,\zeta(z)f(z)}{L^2_\mu \longrightarrow L^2_\mu}$$

This operator is unitary for any measure μ on $\overline{\mathbb{R}}$ and verifies the identities $\sigma_p(S_\mu) = \zeta(\text{mass points of } \mu)$ and $\sigma(S_\mu) = \zeta(\text{supp}\mu)$. The matrix representations of T_μ and S_μ with respect to the related ORF are related to the operator analogs of the new linear fractional transformations ζ_α.

To discuss such operator linear fractional transformations it is convenient to introduce the notation

$$\operatorname{Re} T = \frac{1}{2}(T + T^\dagger), \qquad \operatorname{Im} T = \frac{1}{2i}(T - T^\dagger),$$

for any densely defined operator T on H. The operator linear fractional transformations of interest for ORF on the real line are

$$\zeta_A(T) = \eta_A\, \varpi_A(T)^{-1} \varpi_A^*(T)\, \eta_A^{-1}, \quad \begin{cases} \varpi_A(T) = T - A^\dagger, \\ \varpi_A^*(T) = T - A, \end{cases}$$

$$\tilde{\zeta}_A(T) = \eta_A^{-1} \tilde{\varpi}_A^*(T)\, \tilde{\varpi}_A(T)^{-1} \eta_A, \quad \begin{cases} \tilde{\varpi}_A(T) = 1 - T, \\ \tilde{\varpi}_A^*(T) = \eta_A A \eta_A^{-1} - \eta_A A^\dagger \eta_A^{-1} T, \end{cases}$$

where

$$\eta_A = \sqrt{\operatorname{Im} A}$$

and $A \in \mathbb{B}_H$ is such that $\operatorname{Im} A \geq \varepsilon$ for some positive number ε (in short, $\operatorname{Im} A > 0$), so that η_A is bounded with bounded inverse. When A is normal, as it is the case related to ORF on the real line, $\tilde{\varpi}_A^*$ becomes

$$\tilde{\varpi}_A^*(T) = A - A^\dagger T.$$

ζ_A is a bijection of $\mathbb{U}_H = \{T \in \mathbb{B}_H : \operatorname{Im} T > 0\}$ onto $\mathbb{D}_H$, and $\tilde{\zeta}_A$ is its inverse. To prove this assertion we start showing that ζ_A and $\tilde{\zeta}_A$ map $\mathbb{U}_H$ and $\mathbb{D}_H$ respectively on $\mathbb{B}_H$. The statement for $\tilde{\zeta}_A$ is a consequence of the fact that $\|T\| < 1$ implies $\tilde{\varpi}_A(T)^{-1} \in \mathbb{B}_H$. As for ζ_A, the result follows from the fact that the spectrum of any operator $T \in \mathbb{B}_H$ is included in the closure of its numerical range $\{(x, Tx) : \|x\| = 1\}$. So, if $\operatorname{Im} T > 0$, then $\sigma(T) \subset \mathbb{U}$ and thus $T^{-1} \in \mathbb{B}_H$. In consequence, $\varpi_A(T)^{-1} \in \mathbb{B}_H$ for any $T \in \mathbb{U}_H$ since $\operatorname{Im}(T - A^\dagger) \geq \operatorname{Im} A > 0$.

On the other hand, using the equality $A^\dagger \eta_A^{-2} A = 2iA + A\, \eta_A^{-2} A = A\, \eta_A^{-2} A^\dagger$, we find the identities

$$\begin{aligned} &\varpi_A(T)\, \eta_A^{-1}(1 - \zeta_A(T)\, \zeta_A(T)^\dagger)\, \eta_A^{-1} \varpi_A(T)^\dagger = 4 \operatorname{Im} T, \\ &\tilde{\varpi}_A(T)^\dagger \eta_A^{-1} (\operatorname{Im} \tilde{\zeta}_A(T))\, \eta_A^{-1} \tilde{\varpi}_A(T) = 1 - T^\dagger T, \end{aligned} \tag{46}$$

which prove that ζ_A maps $\mathbb{U}_H$ on $\mathbb{D}_H$ and $\tilde{\zeta}_A$ does the opposite. Moreover, a direct calculation shows that, for any $T \in \mathbb{U}_H$ and any $S \in \mathbb{D}_H$, $S = \zeta_A(T)$ iff $T = \tilde{\zeta}_A(S)$. This completes the proof.

The above arguments can be easily generalized to see that ζ_A extends to a transformation of $\overline{\mathbb{U}}_H = \{T \in \mathbb{B}_H : \operatorname{Im} T \geq 0\}$ onto $\{T \in \overline{\mathbb{D}}_H : 1 \notin \sigma(T)\}$, $\tilde{\zeta}_A$ being its inverse. In consequence, ζ_A maps $\overline{\mathbb{U}}_H \backslash \mathbb{U}_H$ onto $\{T \in \mathbb{T}_H : 1 \notin \sigma(T)\}$ and $\tilde{\zeta}_A$ does the converse. Furthermore, (46) also implies that ζ_A maps the set of bounded self-adjoint operators onto the set of unitary operators whose spectrum does not contain 1.

The above properties are verified in particular by the Cayley transform for operators, since it is given by $\zeta = \zeta_{A=i}$. Indeed, ζ_A is nothing but the composition of the Cayley transform with an operator transformation depending on A which maps onto theirselves $\overline{\mathbb{U}}_H$, $\mathbb{U}_H$ and the set of self-adjoint operators on H. More precisely, taking into account that

$$\begin{aligned} \eta_A^{-1} \varpi_A(T)\, \eta_A^{-1} &= \eta_A^{-1}(T - \operatorname{Re} A)\, \eta_A^{-1} + i, \\ \eta_A^{-1} \varpi_A^*(T)\, \eta_A^{-1} &= \eta_A^{-1}(T - \operatorname{Re} A)\, \eta_A^{-1} - i, \end{aligned}$$

we obtain

$$\zeta_A(T) = \zeta(\eta_A^{-1}(T - \operatorname{Re} A)\, \eta_A^{-1}). \tag{47}$$

It is known that the Cayley transform extends to a bijection between the set of (bounded or unbounded) self-adjoint operators and the set of unitary operators whose point spectrum does not contain 1, so the same holds for ζ_A. The importance of this property is that it permits us to formulate the spectral theory for ORF on the real line, including the case of measures on $\mathbb{R}$ with unbounded support since they are associated with unbounded self-adjoint multiplication operators.

Another advantage of relation (47) is that it expresses ζ_A as a product of two commutative factors. This provides two equivalent representations of ζ_A, namely,

$$\zeta_A(T) = \eta_A(T - A^\dagger)^{-1}(T - A)\, \eta_A^{-1} = \eta_A^{-1}(T - A)(T - A^\dagger)^{-1}\eta_A,$$

giving rise to two expressions for $\tilde{\zeta}_A$ too. From the above result we find that $\zeta_A(T)^\dagger = \zeta_{A^\dagger}(T^\dagger)$ and $\tilde{\zeta}_A(T)^\dagger = \tilde{\zeta}_{A^\dagger}(T^\dagger)$, as in the case of the unit circle.

Finally, if $\mathfrak{I}$ is a hermitian ideal of $\mathbb{B}_H$, similar arguments to those given in the proof of Theorem 6.1 prove that, for any normal operators $A, B \in \mathbb{U}_H$

such that $AB = BA$, the condition $A - B \in \mathfrak{I}$ implies the equivalences

$$\begin{aligned} &T - S \in \mathfrak{I} \;\Leftrightarrow\; \zeta_A(T) - \zeta_B(S) \in \mathfrak{I}, \qquad \forall\, T, S \in \overline{\mathbb{U}}_H, \\ &T - S \in \mathfrak{I} \;\Leftrightarrow\; \tilde{\zeta}_A(T) - \tilde{\zeta}_B(S) \in \mathfrak{I}, \qquad \forall\, T, S \in \overline{\mathbb{D}}_H, \quad 1 \notin \sigma(T) \cup \sigma(S). \end{aligned}$$

The right implication of each case is equivalent to the left implication of the other one due to the fact that ζ_A and $\tilde{\zeta}_A$ are mutually inverse transformations. As we pointed out, when T is unitary the transformation $\zeta_A(T)$ is well defined provided that 1 is not an eigenvalue of T. So, the right implication of the second equivalence can be formulated in a more general context when the operators T, S are unitary. The analogous extension for the right implication of the first equivalence, i.e., the case of T, S unbounded self-adjoint operators, is not possible because we suppose that $\mathfrak{I}$ is an ideal of $\mathbb{B}_H$ (as it is the case for the classes of perturbations usually considered in operator theory). Therefore, we only can assure that

$$\begin{aligned} &T - S \in \mathfrak{I} \;\Leftarrow\; \zeta_A(T) - \zeta_B(S) \in \mathfrak{I}, \qquad \forall\, T, S \text{ self-adjoint}, \\ &T - S \in \mathfrak{I} \;\Rightarrow\; \tilde{\zeta}_A(T) - \tilde{\zeta}_B(S) \in \mathfrak{I}, \qquad \forall\, T, S \text{ unitary}, \quad 1 \notin \sigma_p(T) \cup \sigma_p(S). \end{aligned}$$

These results, although weaker than the previous ones, are enough to apply perturbative techniques to the spectral theory of ORF on the real line, even if the support of the orthogonality measure is unbounded.

With all these operator tools at hand we can develop the spectral theory for ORF on the real line following the same steps as in the case of the unit circle. In fact, the results for the unit circle are formulated throughout the paper in such a way that the translation to the real line is just a matter of changing the meaning of the symbols according to the previous discussion, together with some other obvious modifications. For instance, if μ is a measure on $\overline{\mathbb{R}}$ and $(\phi_n)_{n\geq 0}$ are the ORF associated with an arbitrary sequence $\boldsymbol{\alpha} = (\alpha_n)_{n\geq 1}$ in $\mathbb{U}$, then the sequence $(\chi_n)_{n\geq 0}$ defined by (32), with the new meaning for $\zeta_n = \zeta_{\alpha_n}$, are the ORF associated with $(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$. $(\chi_n)_{n\geq 0}$ is a basis of L^2_μ when the odd and even products $B^o(z)$ and $B^e(z)$ converge to zero for $z \in \mathbb{U}$, but this means now that $\sum_{k=1}^\infty \operatorname{Im} \alpha_{2k-1}/(1 + |\alpha_{2k-1}|^2) = \sum_{k=1}^\infty \operatorname{Im} \alpha_{2k}/(1 + |\alpha_{2k}|^2) = \infty$. Also, if $\boldsymbol{a} = (a_n)_{n\geq 1}$ are the parameters of the recurrence for ϕ_n, the zeros of ϕ_n are the eigenvalues of $\mathcal{U}^{(n)} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{C}_n)$, where $\mathcal{C} = \mathcal{C}(\boldsymbol{a})$ and $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$ as in the case of the unit circle, but $\tilde{\zeta}_{\mathcal{A}_n}$ is the new operator linear fractional transformation given in this section. Notice that $\mathcal{A}_n \in \mathbb{U}_{\mathbb{C}^n}$ and $\mathcal{C}_n \in \mathbb{T}_{\mathbb{C}^n}$

has its eigenvalues in $\mathbb{D}$ because they are the zeros of the n-th OP associated with the parameters $\boldsymbol{a}$. Hence, $\mathcal{U}^{(n)}$ is a well defined matrix of $\overline{\mathbb{U}}_{\mathbb{C}^n} \setminus \mathbb{U}_{\mathbb{C}^n}$, which agrees with the fact that the zeros of ϕ_n lie on $\mathbb{U}$.

Other results for the unit circle can be translated to the real line in a similar way, but two of the main results need a special discussion. The first one concerns the representation of the self-adjoint multiplication operator T_μ for a measure μ on $\mathbb{R}$, and the other one is related to the representation of the self-adjoint multiplication operator $T_{\mu_n^v}$ corresponding to the finitely supported measure μ_n^v associated with the PORF Q_n^v.

Following the same steps as in Theorem 5.1, we would find that, if μ is a measure on $\mathbb{R}$, for any sequence $\boldsymbol{\alpha}$ compactly included in $\mathbb{U}$, the matrix representation of T_μ with respect to the ORF $(\chi_n)_{n\geq 0}$ associated with $(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$ is $\mathcal{U} = \tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$, where $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$, $\mathcal{C} = \mathcal{C}(\boldsymbol{a})$ and $\boldsymbol{a} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. However, since the matrix $\mathcal{C}$ is unitary, we can assure that $\tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ provides a well defined (self-adjoint) operator only when 1 is not an eigenvalue of $\mathcal{C}$. That is, in the case of the real line, the matrix representation $\mathcal{U} = \tilde{\zeta}_A(\mathcal{C})$ is valid provided that $1 \notin \sigma_p(\mathcal{C})$. To understand the meaning of this condition we will relate $\mathcal{C}$ to the matrix representation with respect to $(\chi_n)_{n\geq 0}$ of S_μ. When $1 \notin \sigma_p(\mathcal{C})$ the matrix of $S_\mu = \zeta(T_\mu)$ is $\zeta(\mathcal{U})$, but, as we will see, an expression for the matrix representation of S_μ can be obtained for any measure μ on the real line, even if it has a mass point at ∞. This discussion will lead also to a relation between the operator linear fractional transformations in the real line and the unit circle.

Since we are going to consider at the same time the linear fractional transformations used on the real line and on the unit circle, in what follows we will distinguish between both cases with a superscript $^{\mathbb{R}}$ or $^{\mathbb{T}}$ respectively. Let $A \in \mathbb{U}_H$. Due to the properties of the Cayley transform, $B = \zeta(A) \in \mathbb{D}_H$. A direct computation gives

$$\operatorname{Im} A = (1 - B)^{-1}(1 - BB^\dagger)\,(1 - B^\dagger)^{-1}.$$

Therefore, $\eta_A^{\mathbb{R}} = |\eta_B^{\mathbb{T}}(1 - B^\dagger)^{-1}|$ and, as a consequence of the polar decomposition,

$$\eta_B^{\mathbb{T}}(1 - B^\dagger)^{-1} = U\eta_A^{\mathbb{R}}, \qquad U \text{ unitary.} \tag{48}$$

If we change A by $-A^\dagger$, then B changes to $B^\dagger$, thus,

$$\eta_{B^\dagger}^{\mathbb{T}}(1 - B)^{-1} = V\eta_A^{\mathbb{R}}, \qquad V \text{ unitary.} \tag{49}$$

When A is normal, B is normal too and $\eta_A^{\mathbb{R}} = |1-B|^{-1}\eta_B^{\mathbb{T}}$, so $U = V^\dagger = \xi_B$, where

$$\xi_B = (1-B)|1-B|^{-1}.$$

In the general case, using (48) and (49), we find that

$$\zeta_B^{\mathbb{T}}(\zeta(T)) = U\zeta_A^{\mathbb{R}}(T)V^\dagger,$$

hence

$$\zeta(\tilde{\zeta}_A^{\mathbb{R}}(T)) = \tilde{\zeta}_B^{\mathbb{T}}(UTV^\dagger). \tag{50}$$

Denoting $w = \zeta(z)$ and $S = UTV^\dagger$, a straightforward calculation gives

$$\varpi_A^{*\mathbb{R}}(z) - \varpi_A^{\mathbb{R}}(z)\, T_A^{\mathbb{R}} = \frac{2i}{1-w}\Big(\varpi_B^{*\mathbb{T}}(w) - \varpi_B^{\mathbb{T}}(w)\, S_B^{\mathbb{T}}\Big)(1-B)^{-1}, \tag{51}$$

where $T_A^{\mathbb{R}} = (\eta_A^{\mathbb{R}})^{-1}T\,\eta_A^{\mathbb{R}}$ and $S_B^{\mathbb{T}} = (\eta_B^{\mathbb{T}})^{-1}S\,\eta_B^{\mathbb{T}}$. Since equations (15) and (16) hold for the real line too, the above equality can be written equivalently as

$$z\,\tilde{\varpi}_A^{\mathbb{R}}(T_A^{\mathbb{R}}) - \tilde{\varpi}_A^{*\mathbb{R}}(T_A^{\mathbb{R}}) = \frac{2i}{1-w}\Big(w\,\tilde{\varpi}_B^{\mathbb{T}}(S_B^{\mathbb{T}}) - \tilde{\varpi}_B^{*\mathbb{T}}(S_B^{\mathbb{T}})\Big)(1-B)^{-1}. \tag{52}$$

Using (48) and (49) we obtain $S_B^{\mathbb{T}} = (1-B^\dagger)^{-1}T_A^{\mathbb{R}}(1-B)$. Taking this relation into account, a direct computation yields

$$1-\tilde{\zeta}_B^{\mathbb{T}}(S) = 1-\tilde{\varpi}_B^{*\mathbb{T}}(S_B^{\mathbb{T}})\,\varpi_B^{\mathbb{T}}(S_B^{\mathbb{T}})^{-1} = (\eta_A^{\mathbb{R}})^{-1}(1-T)\,V^\dagger\tilde{\varpi}_B^{\mathbb{T}}(S)^{-1}\eta_{B^\dagger},$$

which implies that

$$1 \in \sigma(\tilde{\zeta}_B^{\mathbb{T}}(S)) \;\Leftrightarrow\; 1 \in \sigma(T), \qquad 1 \in \sigma_p(\tilde{\zeta}_B^{\mathbb{T}}(S)) \;\Leftrightarrow\; 1 \in \sigma_p(T). \tag{53}$$

Assume now that $\boldsymbol{\alpha}$ is compactly included in $\mathbb{U}$ and μ is a measure on $\mathbb{R}$ such that $1 \notin \sigma_p(\mathcal{C})$. From (50) we see that the matrix representation $\zeta(\tilde{\zeta}_{\mathcal{A}}^{\mathbb{R}}(\mathcal{C}))$ of S_μ can be expressed alternatively as $\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\,\mathcal{C}\,\xi_{\mathcal{B}})$, with $\mathcal{B} = \zeta(\mathcal{A})$. Nevertheless, contrary to $\zeta(\tilde{\zeta}_{\mathcal{A}}^{\mathbb{R}}(\mathcal{C}))$, $\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\,\mathcal{C}\,\xi_{\mathcal{B}})$ is always a well defined (unitary) matrix, no matter whether 1 is an eigenvalue of $\mathcal{C}$ or not, because $\xi_{\mathcal{B}}\,\mathcal{C}\,\xi_{\mathcal{B}}$ is unitary and $\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}$ maps unitary operators into unitary operators. Actually, we are going to prove that, if $\boldsymbol{\alpha}$ compactly included in $\mathbb{U}$, $\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\,\mathcal{C}\,\xi_{\mathcal{B}})$ is the matrix representation of S_μ with respect to $(\chi_n)_{n\geq 0}$ for any measure μ on $\overline{\mathbb{R}}$. Following similar arguments to those given in the proof of Theorem 5.1 we find that, for any measure μ on $\overline{\mathbb{R}}$, the ORF $(\chi_n)_{n\geq 0}$ satisfy equation (35) too,

but substituting $\hat{\mathcal{C}}=\mathcal{C}_{\mathcal{A}}^{\mathbb{T}}$ by $\hat{\mathcal{C}}=\mathcal{C}_{\mathcal{A}}^{\mathbb{R}}$, and $\varpi_{\mathcal{A}}^{\mathbb{T}}$, $\varpi_{\mathcal{A}}^{*\mathbb{T}}$ by $\varpi_{\mathcal{A}}^{\mathbb{R}}$, $\varpi_{\mathcal{A}}^{*\mathbb{R}}$ respectively. Applying (51) and using (15) and (16) we conclude that, for $\boldsymbol{\alpha}$ compactly included in $\mathbb{U}$,

$$\begin{pmatrix}\chi_0(z) & \chi_1(z) & \cdots\end{pmatrix}\Big(\zeta(z)-\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\,\mathcal{C}\,\xi_{\mathcal{B}})\Big)=0, \qquad \mathcal{B}=\zeta(\mathcal{A}),$$

which means that $\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\,\mathcal{C}\,\xi_{\mathcal{B}})$ is the matrix of S_μ with respect to $(\chi_n)_{n\geq 0}$. As a consequence of this result and (53), we have the equivalences

$$\begin{gathered}1\in\sigma(\mathcal{C})\;\Leftrightarrow\;1\in\sigma(\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\,\mathcal{C}\,\xi_{\mathcal{B}}))\;\Leftrightarrow\;1\in\sigma(S_\mu)\;\Leftrightarrow\;1\in\zeta(\mathrm{supp}\mu),\\ 1\in\sigma_p(\mathcal{C})\;\Leftrightarrow\;1\in\sigma_p(\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\,\mathcal{C}\,\xi_{\mathcal{B}}))\;\Leftrightarrow\;1\in\sigma_p(S_\mu)\;\Leftrightarrow\;1\in\zeta(\text{mass points of }\mu).\end{gathered}$$

Thus, we have reached the following result.

Theorem 7.1. *Let $\boldsymbol{\alpha}$ be a sequence compactly included in $\mathbb{U}$, μ a measure on $\overline{\mathbb{R}}$ and $\mathcal{C}=\mathcal{C}(\boldsymbol{a})$ with $\boldsymbol{a}=\mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. Then,*

$$1\in\sigma(\mathcal{C})\;\Leftrightarrow\;\infty\in\mathrm{supp}\mu, \qquad 1\in\sigma_p(\mathcal{C})\;\Leftrightarrow\;\infty \textit{ is a mass point of } \mu.$$

Therefore, μ is a measure on $\mathbb{R}$ iff its related sequence $\boldsymbol{a}$ satisfies $1\notin\sigma_p(\mathcal{C})$. Thus, $\mathcal{U}=\tilde{\zeta}_{\mathcal{A}}^{\mathbb{R}}(\mathcal{C})$ provides a well defined matrix representation of T_μ for any measure μ on $\mathbb{R}$. Moreover, the measures on $\mathbb{R}$ with bounded support are characterized by the fact that $\boldsymbol{a}$ is such that $1\notin\sigma(\mathcal{C})$.

In the case of an arbitrary measure μ on $\overline{\mathbb{R}}$, including the possibility of a mass point at ∞, we can study the relation $\mu(\boldsymbol{a},\boldsymbol{\alpha})$ throughout the spectral analysis of the matrix representation $\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\,\mathcal{C}\,\xi_{\mathcal{B}})$ of S_μ or, alternatively, we can deal with a pair of operators. More precisely, relation (52) implies that the spectra of $\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\,\mathcal{C}\,\xi_{\mathcal{B}})$ and the pair $(\tilde{\varpi}_{\mathcal{A}}^{*\mathbb{R}}(\mathcal{C}),\tilde{\varpi}_{\mathcal{A}}^{\mathbb{R}}(\mathcal{C}))$ are related by the Cayley transform, so

$$\mathrm{supp}\mu=\sigma(\tilde{\varpi}_{\mathcal{A}}^{*\mathbb{R}}(\mathcal{C}),\tilde{\varpi}_{\mathcal{A}}^{\mathbb{R}}(\mathcal{C}))=\sigma(\mathcal{C}_o+\mathcal{A}\mathcal{C}_e^\dagger,\mathcal{C}_e^\dagger+\mathcal{A}^\dagger\mathcal{C}_o).$$

Also, the eigenvalues of the pair are the mass points of μ and the eigenvectors of the pair with eigenvalue λ are spanned by $\begin{pmatrix}\chi_0(\lambda) & \cdots & \chi_{n-1}(\lambda)\end{pmatrix}\eta_{\mathcal{A}_n}^{-1/2}$. That is, while the spectral methods that use linear fractional transformations $\tilde{\zeta}_{\mathcal{A}}^{\mathbb{R}}$ of five-diagonal matrices only work for measures on $\mathbb{R}$, their formulation in terms of pairs of band matrices are valid for any measure on $\overline{\mathbb{R}}$.

Similar results hold too for the finitely supported measures associated with the PORF. Given an arbitrary measure μ on $\overline{\mathbb{R}}$, consider the measure

μ_n^v supported on the zeros of the PORF $Q_n^v = \phi_n + v\phi_n^*$, $v \in \mathbb{T}$. As in the case of the unit circle, Q_n^v has n different zeros, but now they lie on $\overline{\mathbb{R}}$. Besides, if $u = \tilde{\zeta}_{a_n}$, the matrix representation $\mathcal{U}^{(n;u)} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{C}_n^u)$ of $T_{\mu_n^v}$ with respect to $(\chi_k)_{k=0}^{n-1}$ is well defined whenever $1 \notin \sigma(\mathcal{C}_n^u)$. Concerning this condition, an analogous argument to that of the measure μ proves that

$$1 \in \sigma(\mathcal{C}_n^u) \;\Leftrightarrow\; \infty \in \operatorname{supp}\mu_n^v,$$

i.e., the matrix representation $\mathcal{U}^{(n;u)}$ of $T_{\mu_n^v}$ is valid for any measure μ_n^v, except for the value $v = -\phi_n^*(\infty)/\phi_n(\infty)$ which locates a zero of Q_n^v at ∞. Nevertheless, analogously to the previous discussion, the spectral interpretation of the PORF in terms of pairs of band matrices given for the unit circle after Theorem 5.9 holds for any PORF on the real line too.

Concerning the applications of the spectral theory for ORF on the real line, from the previous comments we know that, if $\mathfrak{I}$ is an ideal of $\mathbb{B}_{\ell^2}$, for any sequences $\boldsymbol{\alpha}, \boldsymbol{\beta}$ compactly included in $\mathbb{U}$ and any sequences $\boldsymbol{a}, \boldsymbol{b}$ in $\overline{\mathbb{D}}$ such that $1 \notin \sigma_p(\mathcal{C}(\boldsymbol{a})) \cup \sigma_p(\mathcal{C}(\boldsymbol{b}))$,

$$\mathcal{A}(\boldsymbol{\alpha}) - \mathcal{A}(\boldsymbol{\beta}),\, \mathcal{C}(\boldsymbol{a}) - \mathcal{C}(\boldsymbol{b}) \in \mathfrak{I} \;\Rightarrow\; \mathcal{U}(\boldsymbol{a}, \boldsymbol{\alpha}) - \mathcal{U}(\boldsymbol{b}, \boldsymbol{\beta}) \in \mathfrak{I}.$$

This permits us to extend to ORF on $\mathbb{R}$ the applications for ORF on $\mathbb{T}$ discussed in Section 6.

Equation (50) provides a connection between the real line and the unit circle representations. Let $\boldsymbol{\alpha} = (\alpha_n)_{n\geq 1}$ be a sequence compactly included in $\mathbb{U}$, and consider the sequence $\boldsymbol{\beta} = (\beta_n)_{n\geq 1}$ in $\mathbb{D}$ given by $\beta_n = \zeta(\alpha_n)$. Following the previous notation we also have $\alpha_0 = i$, so $\beta_0 = 1$. Consider two sequences $\boldsymbol{a} = (a_n)_{n\geq 1}$ and $\boldsymbol{b} = (b_n)_{n\geq 1}$ in $\mathbb{D}$ related by

$$b_n = \xi_0^2 \xi_1^2 \cdots \xi_{n-1}^2\, a_n, \qquad \xi_n = \frac{1-\beta_n}{|1-\beta_n|}.$$

We have the identities $\mathcal{C}_o(\boldsymbol{b}) = \Lambda^\dagger \xi_{\mathcal{B}}\, \mathcal{C}_o(\boldsymbol{a})\, \Gamma$ and $\mathcal{C}_e(\boldsymbol{b}) = \Gamma^\dagger \mathcal{C}_e(\boldsymbol{a})\, \xi_{\mathcal{B}}\, \Lambda$, where $\mathcal{B} = \zeta(\mathcal{A})$, $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$ and

$$\begin{aligned}
\Gamma &= \begin{pmatrix} \gamma_0 & & \\ & \gamma_1 & \\ & & \ddots \end{pmatrix}, \quad \gamma_0 = 1, \quad \gamma_n = \begin{cases} \overline{\xi}_0^2 \overline{\xi}_2^2 \cdots \overline{\xi}_{n-1}^2 & \text{odd } n, \\ \xi_1^2 \xi_3^2 \cdots \xi_{n-1}^2 & \text{even } n, \end{cases} \\
\Lambda &= \begin{pmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \ddots \end{pmatrix}, \quad \lambda_0 = 1, \quad \lambda_n = \begin{cases} \gamma_{n-1}\xi_n & \text{odd } n, \\ \gamma_{n-1}\overline{\xi}_n & \text{even } n, \end{cases}
\end{aligned} \tag{54}$$

Therefore, $\mathcal{C}(\boldsymbol{b}) = \Lambda^\dagger \xi_{\mathcal{B}}\, \mathcal{C}(\boldsymbol{a})\, \xi_{\mathcal{B}}\, \Lambda$ and, thus, equation (50) implies that

$$\zeta(\boldsymbol{\mathcal{U}}^{\mathbb{R}}(\boldsymbol{a}, \boldsymbol{\alpha})) = \Lambda\, \boldsymbol{\mathcal{U}}^{\mathbb{T}}(\boldsymbol{b}, \boldsymbol{\beta})\, \Lambda^\dagger. \tag{55}$$

This relation can be understood taking into account that the ORF on the real line and the unit circle are related by the Cayley transform. More precisely, $\phi_n(z)$ are ORF on the real line iff $\phi_n(\tilde{\zeta}(z))$ are ORF on the unit circle. If μ is the orthogonality measure on $\overline{\mathbb{R}}$, the corresponding measure ν on $\mathbb{T}$ is given by $\nu(\Delta) = \mu(\tilde{\zeta}(\Delta))$ for any Borel subset Δ of $\mathbb{T}$. Also, the parameters α_n and β_n associated respectively with the poles of $\phi_n(z)$ and $\phi_n(\tilde{\zeta}(z))$ are related by $\beta_n = \zeta(\alpha_n)$. Moreover, ϕ_n satisfies the analogue of recurrence (10) on the real line with coefficients a_n iff $\widehat{\phi}_n = \xi_0^2 \xi_1^2 \cdots \xi_{n-1}^2 \xi_n \phi_n$ satisfies such a recurrence on the unit circle with coefficients $b_n = \xi_0^2 \xi_1^2 \cdots \xi_{n-1}^2 a_n$. If χ_n and $\widehat{\chi}_n$ are the associated ORF (given by the corresponding version of (32) on $\overline{\mathbb{R}}$ and $\mathbb{T}$ respectively), then $\widehat{\chi}_n = \lambda_n \chi_n$ with λ_n as in (54). Therefore, if $\boldsymbol{\alpha}$ is compactly included in $\mathbb{U}$, the matrix representation $\boldsymbol{\mathcal{U}}^{\mathbb{R}}(\boldsymbol{a}, \boldsymbol{\alpha})$ of T_μ with respect to $(\chi_n)_{n \geq 0}$ and the matrix representation $\boldsymbol{\mathcal{U}}^{\mathbb{T}}(\boldsymbol{b}, \boldsymbol{\beta})$ of T_ν with respect to $(\widehat{\chi}_n)_{n \geq 0}$ are related by (55).

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References

[1] N.I. Akhiezer, *Theory of Approximation*, Frederic Ungar Publ. Co., New York, 1956.

[2] N.I. Akhiezer, M.G. Krein, *Some Questions in the Theory of Moments*, Transl. Math. Monographs, Vol.2, AMS, Providence, RI, 1962; Russian original, Kharkov, 1938.

[3] M. Alfaro, *El operador multiplicación en la teoría de polinomios ortogonales sobre la circunferencia unidad*, Proc. II Spanish-Portuguese Mathematical Conference (Madrid, 1973), pp. 13–21, Consejo Sup. Inv. Cient., Madrid, 1977.

[4] T.Ya. Azizov, I.S. Iokhvidov, *Linear operators in spaces with an indefinite metric*, John Wiley & Sons, Ltd., Chichester, 1989.

[5] D. Barrios, G. López, *Ratio asymptotics for polynomials orthogonal on arcs of the unit circle*, Constr. Approx. **15** (1999) 1–31.

[6] I.D. Berg, *An extension of the Weyl-von Neumann theorem to normal operators*, Trans. Amer. Math. Soc. **160** (1971) 365–371.

[7] M.S. Birman, *On existence conditions for wave operators*, Dokl. Akad. Nauk SSSR **143** (1962) 506–509 (Russian).

[8] M.S. Birman, M.G. Krein, *On the theory of wave operators and scattering operators*, Dokl. Akad. Nauk SSSR **144** (1962) 475–478 (Russian).

[9] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, *A density problem for orthogonal rational functions*, J. Comput. Appl. Math. **105** (1999) 199-212.

[10] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, *Orthogonal rational functions*, Cambridge Monographs on Applied and Computational Mathematics, 5, Cambridge University Press, Cambridge, 1999.

[11] M.J. Cantero, L. Moral, L. Velázquez, *Measures and para-orthogonal polynomials on the unit circle*, East J. Approx. **8** (2002) 447–464.

[12] M.J. Cantero, L. Moral, L. Velázquez, *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle*, Linear Algebra Appl. **362** (2003) 29–56.

[13] M.J. Cantero, L. Moral, L. Velázquez, *Minimal representations of unitary operators and orthogonal polynomials on the unit circle*, Linear Algebra Appl. **408** (2005) 40–65.

[14] M.J. Cantero, L. Moral, L. Velázquez, *Measures on the unit circle and unitary truncations of unitary operators*, J. Approx. Theory **139** (2006) 430–468.

[15] Ya.L. Geronimus, *On polynomials orthogonal on the circle, on trigonometric moment problem, and on allied Carathéodory and Schur functions*, Mat. Sb. **15** (1944) 99–130.

[16] L. Golinskii, *Singular measures on the unit circle and their reflection coefficients*, J. Approx. Theory **103** (2000) 61–77.

[17] L. Golinskii, *Operator theoretic approach to orthogonal polynomials on an arc of the unit circle*, Matematicheskaya fizika, analiz, geometriya **7** (2000) 3–34.

[18] L. Golinskii, *On the spectra of infinite Hessenberg and Jacobi matrices*, Matematicheskaya fizika, analiz, geometriya 7 (2000) 284–298.

[19] L. Golinskii, P. Nevai, W. Van Assche, *Perturbation of orthogonal polynomials on an arc of the unit circle*, J. Approx. Theory **83** (1995) 392–422.

[20] W.B. Gragg, *Positive definite Toeplitz matrices, the Arnoldi process for isometric operators, and Gaussian quadrature on the unit circle*, J. Comput. Appl. Math. **46** (1993) 183–198; Numerical Methods of Linear Algebra, pp. 16–32, Moskov. Gos. Univ., Moskow, 1982.

[21] W.B. Jones, O. Njåstad, W.J. Thron, *Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle*, Bull. London Math. Soc. **21** (1989) 113–152.

[22] T. Kato, *Perturbation of continuous spectra by trace class operators*, Proc. Japan. Acad. **33** (1957) 260–264.

[23] M.G. Krein, *On an application of the fixed point principle in the theory of linear transformations of spaces with an indefinite metric*, Uspehi Matem. Nauk (N.S.) **5** (1950), no. 2(36), 180–190 (Russian).

[24] M.G. Krein, *A new application of the fixed-point principle in the theory of operators in a space with indefinite metric*, Dokl. Akad. Nauk SSSR **154** (1964) 1023–1026 (Russian).

[25] M.G. Krein, Yu.L. Šmuljan, *On linear-fractional transformations with operator coefficients*, Mat. Issled **2** (1967), no. 3, 64–96 (Russian); English transl. in Amer. Math. Soc. Transl., Ser. 2, **103** (1974) 125–152.

[26] Y. Last, B. Simon, *The essential spectrum of Schrödinger, Jacobi and CMV operators*, J. Anal. Math. **98** (2006) 183–220.

[27] F. Marcellán, E. Godoy, *Orthogonal polynomials on the unit circle: distribution of zeros*, J. Comput. Appl. Math. **37** (1991) 195–208.

[28] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Vol. 1: Functional Analysis*, Academic Press, New York-London, 1972.

[29] W. Sikonia, *The von Neumann converse of Weyl's theorem*, Indiana Univ. Math. J. **21** (1971/72) 121–124.

[30] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series, American Mathematical Society, Providence, RI, 2005.

[31] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, American Mathematical Society, Providence, RI, 2005.

[32] B. Simon, *CMV matrices: Five years after*, to appear in the Proceedings of the W.D. Evans 65th Birthday Conference, arXiv:math.SP/0603093, 2006.

[33] A.V. Teplyaev, *The pure point spectrum of random polynomials orthogonal on the unit circle*, Soviet Math. Dokl. **44** (1992) 407–411; Dokl. Akad. Nauk SSSR 320 (1991) 49–53.

[34] W.J. Thron, *L-polynomials orthogonal on the unit circle*, Nonlinear numerical methods and rational approximation (Wilrijk, 1987), pp. 271–278, Math. Appl., vol. 43, Reidel, Dordrecht, 1988.

[35] H. Weyl, *Über beschraänkte quadratische Formen, deren Differenz vollstetig ist*, Rend. Circ. Mat. Palermo **27** (1909) 373–392.

[36] D.S. Watkins, *Some perspectives on the eigenvalue problem*, SIAM Rev. **35** (1993) 430–471.